# Galaxy Bias and non-Linear Structure Formation in General Relativity

Tobias Baldauf[a], Uroš Seljak[a,b,c],

Leonardo Senatore[d,e] and Matias Zaldarriaga[f]

(a) Institute for Theoretical Physics, University of Zurich, Zurich, Switzerland

(b) Physics Department, Astronomy Department and Lawrence Berkeley National Laboratory, University of California, Berkeley, CA, USA

(c) Institute for the Early Universe, EWHA Womans University, Seoul, South Korea

(d) Stanford Institute for Theoretical Physics, Stanford University, Stanford, CA, USA

(e) Kavli Institute for Particle Astrophysics and Cosmology, Stanford, CA, USA

(f) School of Natural Sciences, Institute for Advanced Study, Olden Lane, Princeton, NJ 08540, USA

## Abstract

Length scales probed by the large scale structure surveys are becoming closer and closer to the horizon scale. Further, it has been recently understood that non-Gaussianity in the initial conditions could show up in a scale dependence of the bias of galaxies at the largest possible distances. It is therefore important to take General Relativistic effects into account. Here we provide a General Relativistic generalization of the bias that is valid both for Gaussian and for non-Gaussian initial conditions. The collapse of objects happens on very small scales, while long-wavelength modes are always in the quasi linear regime. Around every small collapsing region, it is therefore possible to find a reference frame that is valid for arbitrary times and where the space time is almost flat: the Fermi frame. Here the Newtonian approximation is applicable and the equations of motion are the ones of the standard $N$-body codes. The effects of long-wavelength modes are encoded in the mapping from the cosmological frame to the local Fermi frame. At the level of the linear bias, the effect of the long-wavelength modes on the dynamics of the short scales is all encoded in the local curvature of the Universe, which allows us to define a General Relativistic generalization of the bias in the standard Newtonian setting. We show that the bias due to this effect goes to zero as the square of the ratio between the physical wavenumber and the Hubble scale for modes longer than the horizon, confirming the intuitive picture that modes longer than the horizon do not have any dynamical effect. On the other hand, the bias due to non-Gaussianities does not need to vanish for modes longer than the Hubble scale, and for non-Gaussianities of the local kind it goes to a constant. As a further application of our setup, we show that it is not necessary to perform large $N$-body simulations to extract information about long-wavelength modes: $N$-body simulations can be done on small scales and long-wavelength modes are encoded simply by adding curvature to the simulation, as well as rescaling the time and the scale.

# 1  Introduction and Summary

Large Scale Structure (LSS) surveys are becoming larger and larger, and soon they will be able to probe cosmological modes whose length scale is comparable to the Hubble scale. General Relativistic effects scale as the ratio of the physical wavenumber $k/a$ and the Hubble scale

$$\text{General Relativistic Effects} \sim \left(\frac{Ha}{k}\right)^2 \,, \tag{1}$$

and it is therefore important to take these effects into account in order to be able to interpret next generation of LSS data. All the relativistic effects are basically projection effects relating what happens in one place to what we see: they include such things as lensing, redshift, distortion, gravitational redshift, etc. A consistent derivation of them for dark matter has been recently performed in [1]. Unfortunately we do not observe dark matter directly, but just luminous objects. From the observation of them we are able to reconstruct the dark matter density field by the realization that collapsed objects are biased tracers of the dark matter field. The concept of bias has so far always been defined using the Newtonian approximation that is valid for small length scales. The purpose of this paper is to provide a generalization of this concept that is valid at arbitrary long-wavelengths.

Another reason that motivates us to provide such a generalization is due to the recent observation that non-Gaussianity in the primordial density field can induce a scale dependence in the bias at large wavelengths [2, 3]. In the presence of non-Gaussianities of the local kind, the bias receives a scale dependence that in the Newtonian treatment behaves as

$$\delta_{n_g}(k) = b(k)\delta_m(k) \,, \qquad b_{f_{\rm NL}^{loc.}} \sim b_{f_{\rm NL}^{loc.}=0} \left(1 + f_{\rm NL}^{\rm loc.}\frac{H^2 a^2}{k^2}\right) \,, \tag{2}$$

where $\delta_{n_g}$ is the perturbation to the density of objects, $\delta_m$ the perturbation to the matter density, and $k$ is the wavenumber of the mode, and where we have neglected factors of order unity and the transfer function for simplicity. The important point of this expression is that in the presence of non-Gaussianities that have a non-vanishing squeezed limit, such as the ones of the local kind or the new ones that have been found in the Effective Field Theory of Multifield Inflation [4] with support both on equilateral and squeezed configurations, the bias receives a scale dependence at large scales proportional to $f_{\rm NL}$. This provides an ideal setup for measuring non-Gaussianities in LSS, as the signal is peaked on large scales, where theoretical predictions are under better control. Indeed current limits on $f_{\rm NL}^{\rm loc.}$ obtained from the Sloan Digital Sky Survey (SDSS) data are already competitive with the ones from WMAP [3], and analysis of the bispectrum is expected to be even more promising [5].

An odd feature of (2) is that

$$b_{f_{\rm NL}^{loc.}} \to \infty \quad \text{as} \quad k \to 0 \,. \tag{3}$$

It is equally strange that the standard Gaussian bias does not go to zero as $k \to 0$: one might indeed expect that modes longer than the Hubble scale should have no effects on the local dynamics. Of course, all of these results are due to the fact that we are trusting (2) way into

a regime where it does not apply: as $k/a$ becomes close to $H$, a proper General Relativistic treatment becomes necessary.

The main purpose of this paper is to provide such a General Relativistic generalization of the bias that is valid both in the case of primordial Gaussian and non-Gaussian initial conditions. In doing this, we will also provide a way to understand small $N$-body simulations in the General Relativistic setting, and to show that in order to study the effects of long-wavelength modes, it is not necessary to run large, time consuming, $N$-body simulations. Let us briefly summarize the logic and the main results.

- Cosmological perturbations become non-linear and lead to collapse only on very small scales, where the Newtonian approximation is valid. This suggests that if we insist on describing length scales much smaller than the Hubble scale, then the current Newtonian description is valid.

- Given a perturbed Friedman Robertson Walker (FRW) Universe with fluctuations of arbitrary length scale, it is possible to identify a coordinate frame valid on spatial distances much smaller than the horizon and for an arbitrary amount of time, where the metric appears locally as the one of Minkowski space, with small perturbations of order $(Hx)^2$, $x$ being the spatial distance from the origin. These coordinates represent the inertial frame of a free falling observer, and they are called Fermi coordinates [6]. In the case where the matter is non-relativistic, in this frame the Newtonian approximation is manifest, and we argue that this is the frame where results of small-box $N$-body simulations can be interpreted. We explicitly construct such a reference frame at linear order in the long scale fluctuations for a spherically symmetric configuration of the long-wavelength modes, as this is sufficient for the description of linear bias. Generalizations to different configurations for the long-wavelength modes or to the non-linear level should be straightforward.

- In these coordinates, all the effect of the long-wavelength mode is included in the mapping from the global frame to the Fermi frame, and in the long-wavelength curvature of the local patch. Since for the linear bias we can use spherical symmetry for the long-wavelength modes, the long-wavelength part of the Fermi metric must be equivalent to that of a curved FRW Universe, and therefore all the effect that a long-wavelength mode has on the local dynamics is indeed in the curvature of the local FRW Universe. This is given by

$$\Omega_K \sim \frac{\nabla^2 \zeta(\vec{x}_L, t_L)}{a^2 H^2} \ , \tag{4}$$

where $\zeta$ is the curvature perturbation in comoving gauge, and here for simplicity we have omitted numerical factors given later in the text.

- This allows us to generalize the concept of bias to the General Relativistic setting, by declaring it to be the derivative of the proper number density of objects at a fixed proper

3

time with respect to the curvature of the local Universe:

$$b \sim \frac{1}{n_p}\frac{\partial n_p}{\partial \Omega_K} \quad \Rightarrow \quad \delta_{n_p} \sim b\frac{\nabla^2\zeta}{a^2H^2} + \dots \ , \tag{5}$$

where $n_p$ is the proper number density of objects, $\delta_{n_p}$ their relative overdensity and the dots stand for additional terms coming from various projection effects that we will discuss in the text. Here we have neglected numerical factors. This expression makes sense physically, as for modes much longer than the Hubble scale, $\Omega_K \to 0$, making explicit the General Relativistic statement that metric modes that have no measurable gradients do not affect the local dynamics.

- In presence of primordial non-Gaussianities, the initial conditions for the fluctuations in the Fermi patch can depend on other parameters. In the case of non-Gaussianities of the local kind, initial conditions depend explicitly on $\zeta$, a quantity that has no effect on the local dynamics. In this case, we extend the definition of the bias to include the derivative of the proper number density of objects with respect to the parameter itself. For example, in the case of non-Gaussianities of the local kind, we have

$$b_{f_{\rm NL}^{loc.}} \sim \frac{1}{n_p}\frac{\partial n_p}{\partial \zeta} \quad \Rightarrow \quad \delta_{n_p} \sim b\frac{\nabla^2\zeta}{a^2H^2} + b_{f_{\rm NL}^{loc.}}\zeta + \dots \ , \tag{6}$$

where again the . . . stand for additional terms coming from various projection effects which we will discuss in the text. We see that the relative factor of $k^2$ between the standard bias and the one induced by $f_{\rm NL}$ is preserved in the General Relativistic limit. However, most importantly, the physical effect of long-wavelength fluctuations on the local overdensity does not blow up as $k \to 0$: it is simply the fact that the standard Gaussian effect goes to zero while the non-Gaussian one stays constant.

- Finally we point out that our construction of the local Fermi coordinates shows that it is not strictly required to run time-consuming large-box $N$-body simulations to study the effect of long-wavelength fluctuations: their effect can be simply included by running small-box $N$-body simulations with different cosmological parameters than in the standard cosmology.

Related works on the way to include long-wavelength perturbations inside small-box $N$-body simulations have appeared in [7, 8, 9]. Related work on the way to derive the bias of the local form in the General Relativistic context has appeared in [10, 11, 12, 13] and a connection of the latter to primordial non-Gaussianities of the local form has been made in [14].

## 2 Fermi Coordinates for Perturbed FRW

Given a sufficiently smooth spacetime, it is possible to identify a set of coordinates centered around a timelike geodesic, known as Fermi coordinates [6]. They have two important

properties: the metric is approximately that of Minkowski space, with corrections that start quadratically in the (space-like) geodesic distance from the time-like geodesic taken as the origin, and they are valid in the (spatial) vicinity of the time-like geodesic for all times.

In an FRW spacetime the Hubble expansion appears in the Fermi coordinates as a small correction to the standard dynamics in Minkowski space. This set of coordinates was found for unperturbed FRW first in [46]. Here we are going to provide such a set of coordinates for a linearly perturbed FRW Universe. We will then argue that in this set of coordinates the Newtonian approximation is valid, and that this is actually the frame in which $N$-body simulations are performed. Furthermore, we will provide a mapping from the local Fermi coordinates to the global coordinates of a perturbed FRW, and we will show how simulations have to be performed in order to include the effect of perturbations with wavelengths larger than the box size.

Let us therefore find these coordinates. Let us suppose we have an FRW metric with some linear long-wavelength fluctuations. We start from a perturbed FRW metric in Newtonian gauge:

$$ds^2 = -\left(1 + 2\Phi(\vec{x}_G, t_G)\right) dt_G^2 + a(t_G)^2 \left(1 - 2\Psi(\vec{x}_G, t_G)\right) d\vec{x}_G^2 \ . \tag{7}$$

In app. A we perform the same construction starting from $\zeta$-gauge. Here the subscript $_G$ stands for Global to stress that these coordinates are valid for the entire FRW space. A great simplification comes from the fact that we wish to study the properties of the large scale structures mainly in the regime where the long-wavelength modes are linear: in other words, we are mainly interested in the two-point function of large-scale fluctuations. This has two consequences. First, the behavior of $\Phi$ and $\Psi$ can be found by solving the linear Einstein equations and the linearized equations of motion for matter. For example, we can assume that there is no anisotropic stress at linear level, so that $\Psi = \Phi$. Second, if we wish to compute scalar quantities (as we will wish), we can use superposition principle to restrict ourselves to consider configurations where $\Phi$ is spherically symmetric around one point, let us say the point $\vec{x}_G = 0$. Generalization to the non-linear treatment of $\Phi$ is conceptually straightforward, but computationally not so, and we leave it to future work [1].

In order to find the Fermi coordinates (fig. 1), we can restrict ourselves to the neighborhood of a time-like geodesic. Spherical symmetry suggests to consider the geodesic $\vec{x}_G(t_G) = 0$. If we consider modes whose wavelength is much larger than the region of interest, we can Taylor expand the metric around the origin, and keep only the leading two derivatives. Notice that numerical simulations have to follow dark matter particles, and therefore their region of interest corresponds to scales corresponding to the length traveled by the particles, of the order of the non-linear scale. We obtain:

$$ds^2 \simeq -\left(1 + 2\Phi(\vec{0}, t_G) + \Phi(\vec{0}, t_G)_{,r_G r_G} r_G^2\right) dt_G^2 + a(t_G)^2 \left(1 - 2\Phi(\vec{0}, t_G) - \Phi(\vec{0}, t_G)_{,r_G r_G} r_G^2\right) d\vec{x}_G^2 \ , \tag{8}$$

---

[1] Of course such a non-linear treatment would become much more pressing if we had convincing evidence that the primordial perturbation were non-Gaussian. There is some reason of possible excitement: in the CMB Gaussianity is excluded only at the $2\sigma$ level [15] through the analysis of the three-point function of the orthogonal kind parametrized by $f_{\rm NL}^{\rm orthog.}$ [16].
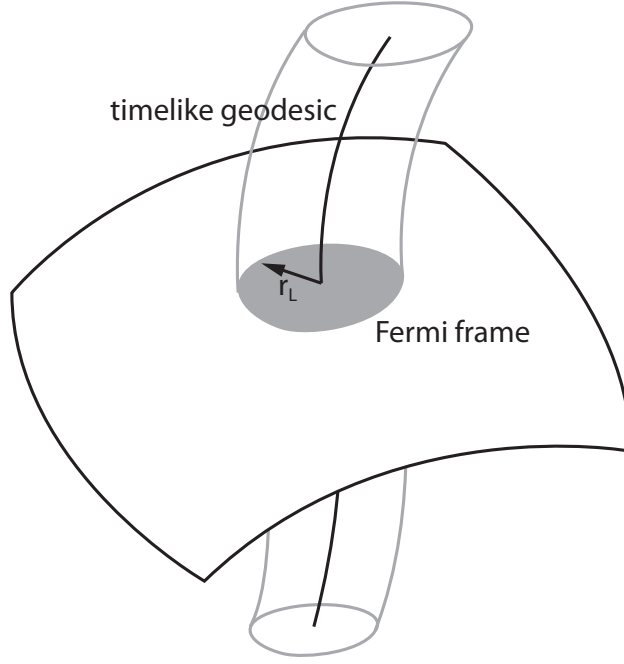
Figure 1: *Fermi Coordinates.*

where $r_G^2 = x_{G,1}^2 + x_{G,2}^2 + x_{G,3}^2$. We can find the coordinates in which the above metric appears in the Fermi way in a simple, but brute force, way that we describe here. A more geometric derivation is presented in app. B. Let us first warm up by considering the case of an unperturbed, curved FRW Universe, whose metric is of the form

$$ds^2 = -dt_G^2 + a(t_G)^2 \frac{d\vec{x}_G^2}{\left[1 + \frac{1}{4} K \, \vec{x}_G^2\right]^2} \ . \tag{9}$$

We consider the curved case here because it will be useful for later purposes. It is easy to check that upon the following change of coordinates, valid at small distances [46]:

$$t_G = t_L - \frac{1}{2} H(t_L) r_L^2 \ , \tag{10}$$

$$x_G^i = \frac{x_L^i}{a(t_L)} \left(1 + \frac{1}{4} H(t_L)^2 r_L^2\right) \ ,$$

where $r_L^2 = x_{L,1}^2 + x_{L,2}^2 + x_{L,3}^2$ and the subscript $_L$ reminds us that these are the Locally valid coordinates, the metric takes the form

$$ds^2 = -\left[1 - \left(\dot{H}(t_L) + H(t_L)^2\right) r_L^2\right] dt_L^2 + \left[1 - \frac{1}{2}\left(H(t_L)^2 + \frac{K}{a(t_L)^2}\right) r_L^2\right] d\vec{x}_L^2 \ . \tag{11}$$

As we had anticipated, for an indefinite amount of time, the metric near the spatial origin is approximately the Minkowski one, with corrections starting at order $r_L^2$ and suppressed by powers of $H \, r_L \ll 1$. So for example this metric is valid for distances smaller than Hubble,

6

but it clearly can include cosmologically interesting length scales such as the non-linear scale where structures form.

To consider now the generic perturbed FRW flat space, let us generalize the change of coordinates as

$$t_G = t_L - \frac{1}{2}H(t_L)r_L^2 - \int_0^{t_L} \Phi(\vec{0}, t')dt' + g_1(t_L)r_L^2 , \tag{12}$$

$$x_G^i = \frac{x_L^i}{a(t_L)}\left(1 + \frac{1}{4}H(t_L)^2 r_L^2 + f_1(t_L) + f_2(t_L)r_L^2\right) ,$$

and let us determine the functions $f_{1,2}$, $g_1$, meant to be first order in the metric fluctuations, by imposing that the metric in the local coordinates is of the Fermi form, with the additional constraint that the spatial part be proportional to $\delta_{ij}$. Notice that we have made the educated guess that at the origin the Local time equals the proper time. We will verify shortly that this is a good guess. After some straightforward algebra, we obtain

$$
\begin{aligned}
t_G &= t_L - \int_0^{t_L} \Phi(\vec{0}, t_L)dt' \\
&\quad - \left(\frac{1}{2}H(t_L) - H(t_L)\Phi(\vec{0}, t_L) - \frac{1}{2}\Phi(\vec{0}, t_L)_{,t_L} - \frac{\dot{H}(t_L)}{2}\int_0^{t_L}\Phi(\vec{0}, t')dt'\right)r_L^2 , \\
x_G^i &= \frac{x_L^i}{a(t_f)}\left[1 + \Phi(\vec{0}, t_L) + H(t_L)\int_0^{t_L}\Phi(\vec{0}, t')dt' + \right. \\
&\quad \left. \frac{1}{4}\left(H(t_L)^2 + H(t_L)\left(H(t_L)^2 - 2\dot{H}(t_L)\right)\int_0^{t_L}\Phi(\vec{0}, t')dt' - H(t_L)^2\Phi(\vec{0}, t_L) - 2H(t_L)\Phi(\vec{0}, t_L)_{,t_L}\right)r_L^2\right] .
\end{aligned}
\tag{13}
$$

Let us recall the common definition of the comoving-gauge curvature perturbation $\zeta$

$$\zeta(\vec{x}_G, t_G) = -\Phi(\vec{x}_G, t_G) + \frac{H(t_G)^2}{\dot{H}(t_G)}\left(\Phi(\vec{x}_G, t_G) + \frac{\dot{\Phi}(\vec{x}_G, t_G)}{H(t_G)}\right) , \tag{14}$$

and the fact that this is constant for adiabatic fluctuations and for wavelengths longer than the sound horizon:

$$\dot{\zeta}(\vec{x}_G, t_G) = \frac{H(t)}{\dot{H}(t)}\left[\ddot{\Phi}(t) + \left(H(t) - \frac{\ddot{H}(t)}{\dot{H}(t)}\right)\dot{\Phi}(t) + \left(2\dot{H}(t) - \frac{\ddot{H}(t)H(t)}{\dot{H}(t)}\right)\Phi(t)\right] = 0 , \tag{15}$$

where the dot stays for derivative with respect to the time variable. This implies that we can write $\zeta$ as

$$\zeta(t) = -\Phi(t) - H(t)\int_0^t dt'\, \Phi(t') , \quad \Rightarrow \quad \dot{\Phi}(t) + H(t)\Phi(t) + \dot{H}(t)\int_0^t dt'\Phi(t') = 0 , \tag{16}$$

and therefore we can simplify the former expressions to get

$$
\begin{aligned}
t_G &= t_L - \int_0^{t_L}\Phi(\vec{0}, t_L)dt' - \frac{1}{2}H(t_L)\left(1 - \Phi(\vec{0}, t_L)\right)r_L^2 , \\
x_G^i &= \frac{x_L^i}{a(t_f)}\left[1 + \frac{H(t_L)^2}{4}r_L^2\right]\left(1 - \zeta(\vec{0}, t_L)\right) .
\end{aligned}
\tag{17}
$$

7

The resulting metric is of the form

$$
\begin{aligned}
ds^2 = & -\left[1 - \left\{\dot{H}(t_L) + H(t_L)^2 - 2\left(H(t_L)^2 + \dot{H}(t_L)\right)\Phi(\vec{0}, t_L) - 3H(t_L)\Phi(\vec{0}, t_L)_{,t_L}\right.\right. \\
& \left.\left. -\Phi(\vec{0}, t_L)_{,t_L t_L} - \left(2H(t_L)\dot{H}(t_L) + \ddot{H}(t_L)\right)\int_0^{t_L}\Phi(\vec{0}, t')dt' - \frac{\Phi(\vec{0}, t_L)_{,r_G r_G}}{a(t_L)^2}\right\}r_L^2\right]dt_L^2 + \\
& + \left[1 - \left\{\frac{H(t_L)^2}{2} - H(t_L)^2\Phi(\vec{0}, t_L) - H(t_L)\Phi(\vec{0}, t_L)_{,t_L}\right.\right. \\
& \left.\left. -H(t_L)\dot{H}(t_L)\int_0^{t_L}\Phi(\vec{0}, t')dt' + \frac{\Phi(\vec{0}, t_L)_{,r_G r_G}}{a(t_L)^2}\right\}r_L^2\right]d\vec{x}_L^2 .
\end{aligned}
\tag{18}
$$

which is valid without assuming that $\zeta$ is constant. If we use that $\zeta$ is indeed constant outside of the sound horizon, the metric simplifies to

$$
\begin{aligned}
ds^2 = & -\left[1 - \left(\dot{H}(t_L) + H(t_L)^2 - \frac{\Phi(\vec{0}, t_L)_{,r_G r_G}}{a(t_L)^2}\right)r_L^2\right]dt_L^2 \\
& + \left[1 - \left(\frac{H(t_L)^2}{2} + \frac{\Phi(\vec{0}, t_L)_{,r_G r_G}}{a(t_L)^2}\right)r_L^2\right]d\vec{x}_L^2 .
\end{aligned}
\tag{19}
$$

The above metric represents the description of a perturbed FRW Universe on scales much smaller than the typical length scale of the perturbations. For this reason, in the presence of adiabatic perturbations whose wavelength is longer than the sound horizon, it has to be equivalent to the local version of an FRW metric, as represented in the local coordinates of (11). This is indeed due to Birkhoff theorem. This is in fact true: upon identification of an effective local expansion rate $H_L(t_L)$ and of an effective curvature $K_L$ given by

$$
H_L(t_L) = H(t_L) + \frac{1}{H(t_L)a(t_L)^2}\left(\Phi(\vec{0}, t_L) + \zeta(\vec{0}, t_L)\right)_{,r_G r_G} ,
\tag{20}
$$

$$
K_L = 2\left[\Phi(\vec{0}, t_L) - \frac{H(t_L)^2}{\dot{H}(t_L)}\left(\Phi(\vec{0}, t_L) + \frac{\Phi(\vec{0}, t_L)_{,t_L}}{H(t_L)}\right)\right]_{,r_G r_G} = -\frac{2}{3}\nabla_G^2\zeta(\vec{0}, t_G) ,
$$

where $H_L(t_L) = \dot{a}_L(t_L)/a_L(t_L)$, the metric (18) takes the form of the curved unperturbed FRW Universe in (11) with the simple replacement $a \to a_L$, $K \to K_L$ [2]. In this case, the local curvature $K_L$ is proportional to the Laplacian of the curvature perturbation usually

---

[2]It might be useful to notice that the curvature perturbation $K = -\frac{2}{3}\nabla^2\zeta$ can be expressed in terms of the matter density perturbation in comoving gauge $\delta_l^{(com)}$ as

$$
K = \left(\Omega_{m,0} + \frac{2}{3}f_0\right)H_0^2\delta_{l,0}^{(com)} ,
\tag{21}
$$

where $H_0$ is the Hubble parameter at the present time, $\Omega_{m,0}$ is the fraction of energy in matter at present time, and $f = \frac{\partial \log D}{\partial \log a}$ with $D$ being the growth factor such that $\delta^{(com)}(t) = D(t)\delta_0^{(com)}$. The subscript 0 is used for quantities evaluated at redshift zero.

denoted by $\zeta$, and is thus constant in time. $H_L$ follows the normal Friedmann equations for a curved FRW.

In summary, we have been able to see that an FRW Universe with a linear adiabatic perturbation whose wavelength is longer than the sound horizon can be described, locally, by a metric that is very close to the Minkowski one, and is actually equivalent to one of a curved FRW Universe. The assumption of adiabaticity and that the wavelength of the mode is longer than the sound horizon is necessary in order for the curvature of the Universe to be constant in time: it is only in this case that there is one single local history for the Universe, which implies that the long-wavelength mode at linear level can be completely re-absorbed into the curvature of a local FRW Universe. In practice, this implies that our method of dealing with long wavelength perturbations is applicable to adiabatic long-wavelength fluctuations in the case where the speed of sound of the fluctuations is very small. This includes a Universe filled with dark matter and a cosmological constant, or with dark matter and quintessence with a very small speed of sound as the models studied in [17], while it does not apply to models with quintessence with non-vanishingly small speed of sound.

Although for some questions one can restrict to the case of spherical symmetry, this is not possible in general. The Fermi coordinates exist also in the absence of spherical symmetry. In app. C we present the form of the Fermi coordinates starting in Newtonian gauge with a plane wave perturbation.

## 2.1 A Simple Check

It is worth to show explicitly how our procedure works in a practical example, where the long-wavelength fluctuation is short enough to allow for a Newtonian treatment. Since we just said that the effect of a long mode can be re-absorbed in a curvature of the background (at linear level and after using superposition principle), this suggests that we should be able to re-derive the growth function at second order for short wavelength fluctuations in the presence of longer, spherically symmetric fluctuations as derived in [39] in the standard perturbation theory approach. Here instead we derive it from the growth of modes in a curved Universe. Working only in the limit where all the modes are describable within the Newtonian approximation and working in Einstein-de-Sitter space, this calculation is carried out in app. D, and here we summarize the main results.

The evolution of short modes in the effective curved Universe is related to the short modes in an flat Universe $\delta_{s,\text{flat}}$ as

$$\delta_s(\vec{x}) = \delta_{s,\text{flat}}(\vec{x})\left(1 + \frac{34}{21}\delta_l(\vec{x})\right) \ , \tag{22}$$

where we have restricted ourselves to the matter only Einstein-de-Sitter Universe. In Fourier space we get

$$\delta_s(\vec{k_s}) = \delta_{s,\text{flat}}(\vec{k_s})\left(1 + \frac{34}{21}\delta_l(k_l)\right) \ , \tag{23}$$

where we have assumed that the long mode is peaked at one frequency $k_l$.

9

Exactly the same expression can be computed in standard perturbation theory [39] as the coupling between an general short and a spherically symmetric long mode $\delta(k_l)$ leading to

$$\delta_s^{(2)}(\vec{k}_s) = \delta_s^{(1)}(\vec{k}_s) + \int \frac{d\Omega_l}{4\pi} F_2(\vec{k}_s, \vec{k}_l) \delta_s^{(1)}(\vec{k}_s) \delta_l^{(1)}(k_l) = \delta_s^{(1)}(\vec{k}_s) \left( 1 + \frac{34}{21} \delta_l^{(1)}(k_l) \right) \ . \qquad (24)$$

We see that for long modes sufficiently far within the horizon so that a Newtonian treatment is possible, the two expressions agree.

# 3    The Coordinate Frame of $N$-body Simulations

Usually, $N$-body simulations are performed on very small scales compared to the Hubble scale, and no hint is usually given onto in what gauge the calculation is actually performed. Further, the equations that are solved in the simulations are not even the General Relativistic equations, but the Newton's equations, where all the General Relativistic effects are neglected [3].

Of course, there is a good reason for this. Usually simulations are performed in boxes which are much smaller than the Hubble scale. Since all General Relativistic effects, from the corrections to Newton's equation to the specification of the coordinate frame, scale proportionally to $(Ha)/k$, these effects are usually negligible. We begin to need to worry when the box size of the simulations becomes larger and larger, and reaches the Hubble scale. At this point, at least naively, we have to modify our codes to include the General Relativistic equations, choose some gauge in which to perform the calculation, take care of what is actually the observable quantity that needs to be computed. This is in fact different from $\delta\rho_m/\rho_m$, $\rho_m$ being the matter density, as recently stressed in [1], due to lensing and redshift distortion effects. But doing all of this may seem a bit too much: at the end of day, we know that large scales evolve linearly, and it is only scales much smaller than the horizon that become non-linear and require $N$-body simulations. Further, if the sound horizon is much smaller than the Hubble scale, local *dynamics* does not really probe long distances, but it only probes distances of the order of the mean free path of the particles, which is the non-linear scale, and so it should not be affected by General Relativistic effects. On small scales, we should be able to apply the Newtonian approximation, and so our way of doing simulations should be fine to describe the small scale non-linearities. There seems to be a tension between including long wavelength fluctuations in the simulations, and the fact that the non-linearities occur just on small scales.

This tension has been solved in a recent paper [18], where it was shown how, exploiting the above facts, it is possible to re-interpret the results of current Newtonian $N$-body simulations directly in the General Relativistic context, by providing a mapping between the results of $N$-body simulations and the fluctuations in a specific gauge valid at arbitrary length scales. The

---

[3]The fact that usually the spatial coordinates are rescaled by a time-dependent factor equal to the scale factor, using the so-called comoving coordinates, should not be misleading: that is just a convenient change of variables for the same equations, which are still just the Newtonian ones.

only mistakes in this procedure are suppressed by powers of $(v/c) \ll 1$, with no corrections of the form $(Ha)/k$.

We are now going to argue that this same tension between $N$-body simulations and General Relativistic effects can be resolved in yet another way, by simply stating that in order to include long wavelength modes into the simulations, it is not necessary to make large-box simulations, but it is simply necessary to perform small-box simulations, in slightly curved backgrounds. The results obtained from the small scale simulations can then be reinterpreted as results obtained in local patches of the whole Universe. We will provide such a mapping [4].

Let us see how this works by showing that simulations can be interpreted in the General Relativistic context as nothing but solving the Einstein equations in the frame defined by the local coordinates (13), where the metric has the form (11) with the scale factor, the Hubble rate and the curvature as given by (20). If we now add short scale perturbations $\delta\Phi$ to the metric we have:

$$
\begin{aligned}
ds^2 &= -\left[1 - \left(\dot{H}_L(t_L) + H_L(t_L)^2\right) r_L^2 + 2\delta\Phi(\vec{x}_L, t_L)\right] dt_L^2 \\
&\quad + \left[1 - \frac{1}{2}\left(H_L(t_L)^2 + \frac{K}{a_L(t_L)^2}\right) r_L^2 - 2\delta\Phi(\vec{x}_L, t_L)\right] d\vec{x}_L^2 \,.
\end{aligned}
\tag{25}
$$

As we write down the Einstein equations for a Universe of dark matter particles plus a cosmological constant, where the perturbations are non-relativistic, we immediately realize that in the above metric the Newtonian approximation is valid: the metric looks like Minkowski with just small corrections, and the system is non-relativistic. Straightforward algebra then shows that the Einstein equations take the form of the simple Poisson equation

$$
\nabla^2 \delta\Phi(\vec{x}_L, t_L) = 4\pi G \delta\rho(\vec{x}_L, t_L) \,,
\tag{26}
$$

while the geodesic equation for the dark matter particles takes the form

$$
\ddot{\delta\vec{x}}(t_L) + 2H_L(t_L)\dot{\delta\vec{x}}(t_L) = -\vec{\nabla}\delta\Phi(\vec{x}(t_L), t_L) \,.
\tag{27}
$$

In obtaining the above two equations, we have done several approximations and definitions that require explanation. We have defined

$$
\delta\rho(\vec{x}_L, t_L) = \rho(\vec{x}_L, t_L) - \frac{3}{8\pi G}\left(H_L(t_L)^2 + \frac{K}{a_L(t_L)^2} - \frac{\Lambda}{3}\right) \,, \qquad \dot{\vec{x}}(t_L) = H_L(t_L)\vec{x} + \dot{\delta\vec{x}}(t_L) \,,
\tag{28}
$$

where here we decided to focus on a $\Lambda$CDM Universe, thought we stress that trivial generalization of our formulas apply to the case of clustering dark energy [17]. Notice that the unperturbed velocity is nothing but the Hubble flow as seen at small distances from the origin.

---

[4]The statement that in order to include large scale modes into small-box simulations one should include curvature and a rescaling of the coordinates has been already given in [21] and then more properly in [22]. However, a mapping from the frame of the simulations to the global frame had not been given, nor, it seems to us, a clear derivation has been presented. Further, all the statements in [21, 22] are not in the General Relativistic context. All of this becomes important if we are dealing with modes comparable to Hubble size.

Then, we have expanded in perturbations by applying the Newtonian approximation: i.e., we have counted the perturbations in powers of $\delta\Phi \sim v^2$, where $\vec{v} = \dot{\vec{x}}(t_L)$, and taken the linear equations in these perturbations. Notice that this amounts to taking the leading terms also in $r_L^2$ in the Einstein equations, while we have not expanded in $\delta\rho/\rho$. The fact that these approximations are justified can be checked a-posteriori, but will become clear in the next paragraph.

In fact, eqs. (26) and (27) are *exactly the same equations* that are solved in $N$-body numerical simulations. This tells us two important things. First, that the Newtonian approximation is indeed justified. Second, most importantly, we *now know* how to interpret the above equations in a General Relativistic setting: they are the equations for a local patch described by the local frame. Thanks to the change of coordinates in (13), we can interpret the results of the $N$-body simulations as points in the full manifold of the spacetime (let us say for example as described in standard Newtonian gauge).

The presence of a long-wavelength mode affects the result of the $N$-body simulations in two different ways: first it affects the mapping from the global to the local coordinates in (13), second it affects the evolution of the short modes by adding a small curvature (20) to the effective local FRW Universe.

In summary, what we found can be synthesized by stating the following simple procedure for performing $N$-body simulations that include large scale fluctuations. Simulations are to be thought of as computing the gravitational structures in the local frame defined by the change of coordinates (13). In the presence of a long-wavelength mode, simulations should be performed in a curved (background) Universe where the curvature is given by (20) [5]. Any scalar quantity measured in the simulations, let us say the proper number density of halos of a given mass, should be intepreted as given at this time:

$$N-\text{body Simulations} \quad \rightarrow \quad n_p^L(\vec{x}_L, t_L; \Omega_K(\zeta)) \ , \tag{29}$$

where the explicit dependence on $\zeta$ comes from the curvature, and the superscript $^L$ reminds us that the output of the $N$-body simulations is to be interpreted as given in Local coordinates. From the mapping (13), we then finally get the value in the set of coordinates that are globally valid, for example in Newtonian gauge:

$$n_p^G(\vec{x}_G, t_G; \zeta) = n_p^L\left(\vec{x}_L(\vec{x}_G, t_G), t_L(\vec{x}_G, t_G); \Omega_K(\zeta)\right) \ , \tag{30}$$

where the superscript $^G$ reminds us that this quantity is defined in global coordinates valid everywhere, and we have used that the proper number density is a scalar.

Finally, we should comment on the initial conditions for the patches corresponding to the regions of space simulated in the $N$-body simulations. In the case of Gaussian initial conditions (we will comment on non-Gaussian initial conditions in the next section), it will turn out that to a very good approximation the initial power spectrum, expressed in terms of the local coordinates, should be the same as it would be in the absence of the long-wavelength

---

[5]As we stressed, the same approach can be generalized to include perturbations at non-linear level and to compute non-scalar quantities: in this case the local patch will not evolve as a curved FRW.

mode. In order to understand the reason of this, it is useful to express the global metric in the comoving ($\zeta$) gauge which is comoving with the density perturbations (see appendix F). In this gauge, for adiabatic initial conditions, and for modes that are far outside of the sound horizon, the metric takes the form

$$ds^2 = -dt^2 + a^2 e^{2\zeta} d\vec{x}^2 \ . \tag{31}$$

Let us decompose the fluctuation $\zeta$ in a long-wavelength and a short-wavelength component $\zeta_l + \zeta_s$, where $_l$ stays for long, and $_s$ stays for short. Let us assume for the moment that the long-component is on scales longer than the sound-horizon. This means that it entered the Hubble scale after matter-radiation equality. In this case, $\zeta_l$ is constant in time. The property of the exponential is such that $\text{Exp}(\zeta) = \text{Exp}(\zeta_l) + \text{Exp}(\zeta_s)$, which implies that in the limit in which we can neglect completely the gradients of $\zeta_l$, $\zeta_l$ can be re-absorbed in a constant rescaling of the scale factor, and is therefore unobservable. This implies that the local physics (from matter radiation equality to recombination and so on) happens in exactly the same way as if the long mode was absent. As we learned in the former section, when we consider gradients of $\zeta_l$, the leading effect of the long mode is to induce a curvature for the local Universe, which clearly affects the local evolution. So, the initial power spectrum of the short scales modes is the one that is obtained in a curved FRW Universe where the curvature is given by the Laplacian of $\zeta_l$ as in eq. (20). In practice, this means that we should run numerical codes as CMBFAST [23] or CAMB [24], run them with the relevant curvature of the Universe, and, after a rescaling by the scale factor, simply interpret the output as in local coordinates. In reality, it is not even necessary to obtain the power spectrum in such a curved Universe as it is easy to realize that the initial curvature is negligible. The relevance of the curvature scales as $\nabla_G^2 \zeta_l/(a_G^2 H_L^2) \propto 1/\dot{a}_L^2$, and therefore it becomes irrelevant in the past. In practice, neglecting the effect of the initial curvature amounts to neglecting terms of order $\nabla_G^2 \zeta_l(t_{L,in})$, where $t_{L,in}$ is the initial time of the $N$-body simulation. When we later define the bias we will define it as the coefficient of proportionality between the local number density and $\nabla_G^2 \zeta_l(t_{L,obs})$, where $t_{L,obs}$ is the time of observation. The effect of the initial term scales as $\dot{a}_L(t_{L,obs})^2/\dot{a}_L(t_{L,in})^2$ and gives a negligible contribution to the bias if the initial time of the $N$-body simulation is early enough. In practice, this is the simple fact that the curvature is irrelevant at early times. This implies that, for long modes that entered the horizon during matter domination, the initial conditions for the simulations are equivalent to the ones in an unperturbed Universe.

The situation becomes slightly more complicated for long wavelength modes that enter the horizon during radiation domination. In this case, there is a window of time from horizon re-entry to matter-radiation equality during which $\zeta_l$ depends on time. This means that the mode in this case can not simply be interpreted as a rescaling of $a$ and an additional curvature term. In this case gradients of the long fluctuation are relevant, as the mode travels approximately an Hubble horizon in an Hubble time. In order to evaluate the effect of the long mode on the short scale power, one should then solve the non-linear equations that couple $\zeta_s$ and $\zeta_l$, along the line of what done in [19]. However, we can argue that this effect is negligible. The biasing of structures as due to a long wavelength mode is an intrinsically

non-linear effect, and it therefore receives most of its contribution from late times, as density perturbations become closer and closer to being non-linear. In perturbation theory, it is straightforward to realize that neglecting the non-Gaussianities of the initial conditions set up at a time parameterized by $a_{in}$ amounts to neglecting a non-Guassianity of the matter fields at a late time parametrized by $a_{obs}$ that is of the order of $a_{in}/a_{obs}$. This is equivalent to the order of the relative error in the bias we have if we neglect the non-Gaussianity in the initial conditions. By taking the initial conditions to be early enough, we can make this error small enough. Given the fact that it is quite hard to measure the bias to great precision, the initial condition can be set up at a reasonably late time.

Let us summarize the discussion about the initial conditions. Concerning modes that entered the horizon during matter domination, one can simply take the power spectrum in local coordinates as in an unperturbed FRW Universe. Concerning modes that entered the horizon during radiation domination, one should take non-Gaussian initial conditions that can be estimated in perturbation theory as for example in [19]; however, their effect is likely to be negligible. The procedure we have outlined in this section enables to extract information about very long wavelength modes without practically modifying the $N$-body codes, and without having to run very large and time-consuming simulations. This should give a valid description for certain questions, such as the halo mass function, where spherical symmetry that we assumed to derive Fermi coordinates is likely to be valid (see app. C for a plane wave case). In app. B.3 and B.4 we give a detailed recipe for how to run a $N$-body simulation given the cosmological parameters and the amplitude of the long-wavelength mode.

# 4 Bias in General Relativity and its Scale Dependence

As an application of our technique we will derive an expression for the bias that is valid in the General Relativistic setting. As it has been recently noted in [2, 3] in the case of the local kind of non-Gaussianities parametrized by the parameter $f_{\rm NL}^{\rm loc.}$, the bias on large scales (as usually measured with respect to to the local matter overdensity) receives a contribution that is scale dependent, proportional to $1/k^2$, where $k$ is the wavenumber of the long-wavelength mode, proportional to $f_{\rm NL}^{\rm loc.}$. The same is expected to be true for the new non-Gaussian shapes that have been found in the Effective Theory of Multifield Inflation [4] (a generalization of the Effective Field Theory of inflation [20]) that have support both in the equilateral and in the squeezed limit. These results were derived in the Newtonian approximation, and here we will derive their generalization for wavelengths comparable or longer than the horizon.

## 4.1 Gaussian Bias

If we consider surveys that are comparable to the horizon scale, then relativistic effects become important and one needs to be very careful in defining observables. We do not directly observe the proper number of galaxies at a given point $n_p(t_G, \vec{x}_G)$ because the photons are deflected and redshifted on their way from the source galaxy to the observer.

What we can do, is count the number of galaxies in bins of angle and redshift. We will refer to the observed number density of galaxies, i.e., the number of galaxies divided by the observed volume, as $n_{obs}(z, \theta, \phi)$. Here $z$ is the observed redshift of the bin, and the tuple $(\theta, \phi)$ represents the observed angular position.

The observed position $(z, \theta, \phi)$ corresponds to a set of global coordinates $(t_G, \vec{x}_G)$. Here we make use of the fact that a spacetime point can be described in in different coordinate systems and that global, local and observed coordinates are just three choices of such a coordinate frame that describe the same point. Thus the global coordinates are a function of the observed coordinates

$$(t_G, \vec{x}_G) = (t_G(z, \theta, \phi), \vec{x}_G(z, \theta, \phi)) \ , \tag{32}$$

and since the proper number density $n_p$ is a scalar, i.e., a function of the point rather than its coordinates, we have

$$n_p(z, \theta, \phi) = n_p\big(t_G(z, \theta, \phi), \vec{x}_G(z, \theta, \phi)\big) \ . \tag{33}$$

To compute the observed number density $n_{obs}(z, \theta, \phi)$ we need to model both the proper density of objects $n_p$ and the mapping between proper and observed coordinates. Let us start with the proper number density.

### 4.1.1   Proper and Observed Number Density

We have argued that in presence of long wavelength modes, the local inertial frame corresponds to a homogeneous curved FRW Universe. As a result the proper number density of galaxies at the spacetime point is given by the number density in the effective curved Universe. We will denote this number $n_p(t_L; \Omega_K)$. The time argument $t_L$ stresses the fact that the proper time of the free falling observer is in general different from the global coordinate time.

We have:
$$n_p(z, \theta, \phi) = n_p\big(t_L\big(t_G(z, \theta, \phi), \vec{x}_G(z, \theta, \phi)\big); \Omega_K\big) \ , \tag{34}$$

where $t_L(t_G, \vec{x}_G)$ denotes the time in the Fermi frame centered at $(t_G, \vec{x}_G)$ and $\Omega_K$ is the curvature associated with the long wavelength mode. To evaluate this expression we need to compute the relation between $(z, \theta, \phi)$ and $t_L$. We can split this relation in two parts.

First we can relate $(z, \theta, \phi)$ to the global coordinates. As shown in [1] there is a lapse between the coordinate redshift $1 + z_G = 1/a_G(t_G)$ and the observed redshift $z$

$$z - z_G = (1 + z_G)\delta z_{G \to z} \tag{35}$$

where $\delta z_{G \to z}$ is given in app. F. We also need to relate the global time coordinate to the time in the Fermi frame at the origin (see eq. 13),

$$t_L(t_G, \vec{x}_G) = t_G + \delta t_{G \to L}(t_G, \vec{x}_G). \tag{36}$$

The time shift between the global and the local coordinates is the difference between the global coordinate-time and the proper-time. In Newtonian gauge we have

$$\delta t_{G\to L}(z,\theta,\phi) = t_L\big(t_G(z,\theta,\phi)\big) - t_G(z,\theta,\phi) = \int_0^{t_G(z,\theta,\phi)} \Phi(t_G'(z,\theta,\phi), \vec{x}_G(z,\theta,\phi)) dt_G' \ , \quad (37)$$

$$= -\frac{1}{H(z)} \left[ \zeta\big(t_G(z,\theta,\phi), \vec{x}_G(z,\theta,\phi)\big) + \Phi\big(t_G(z,\theta,\phi), \vec{x}_G(z,\theta,\phi)\big) \right] \ .$$

We can now expand eq. (34) to first order in the perturbations to obtain,

$$n_p(t_L; \Omega_K) = n_p(t_L, \Omega_K = 0) \left[ 1 + \frac{1}{\bar{n}_p} \frac{\partial n_p}{\partial \Omega_K} \Omega_K \right] \ , \quad (38)$$

where $\bar{n}_p$ is the unperturbed number density at the redshift of observation. Doing so, we have performed a split into background and perturbation such that $n_p(t_L, \Omega_K = 0)$ is not a scalar but a function of its time argument. Thus

$$n_p(z,\theta,\phi) = n_p(t_G, \Omega_K = 0) \left[ 1 + \frac{1}{\bar{n}_p} \frac{\partial n_p}{\partial \Omega_K} \Omega_K + \frac{\partial \log \bar{n}_p}{\partial t} \delta t_{G\to L} \right] \quad (39)$$

$$= n_p(z_G, \Omega_K = 0) \left[ 1 + \frac{1}{\bar{n}_p} \frac{\partial n_p}{\partial \Omega_K} \Omega_K + \frac{\partial \log \bar{n}_p}{\partial \log(1+z)} \delta z_{G\to L} \right] \ , \quad (40)$$

where we have rewritten the prefactor and the time shift in terms of the global redshift $z_G$, which is possible since there is a one-to-one relationship between redshift and time in the auxiliary background Universe that can be translated into a relation between $\delta t_{G\to L}$ and $\delta z_{G\to L}$

$$\delta t_{G\to L}(z,\theta,\phi) = -\frac{z_G(t_L) - z_G(t_G)}{H(z)(1+z)} = -\frac{\delta z_{G\to L}}{H(z)} \ . \quad (41)$$

When calculating spherical averages, the observed redshift $z$ is fixed while coordinate redshift $z_G$ and global time $t_G$ vary. As we will see shortly, it is beneficial to evaluate the prefactor at $z = z_G + \delta z_{G\to z}$

$$n_p(z,\theta,\phi) = n_p(z, \Omega_K = 0) \left[ 1 + \frac{1}{\bar{n}_p} \frac{\partial n_p}{\partial \Omega_K} \Omega_K + \frac{\partial \log \bar{n}_p}{\partial \ln(1+z)} \left( \delta z_{G\to L} - \delta z_{G\to z} \right) \right] \ . \quad (42)$$

We can now define the bias as

$$b_{\Omega_K}(t) = -\frac{1}{\bar{n}_p} \frac{\partial n_p}{\partial \Omega_K}, \quad (43)$$

and use that $\Omega_K(t) = 2\nabla_G^2 \zeta/(3a^2 H^2)$ in eq. (39). We discuss the relation between this definition of the bias and the standard one in the Newtonian approximation in the next section.

### 4.1.2 Volume Distortion

Finally, to compute $n_{obs}(z,\theta,\phi)$ we need to take into account the distortions in the volume induced by the mapping between $(z,\theta,\phi)$ and the local frame. These geometric factors were

recently derived at linear level in [1]. We denote $V_p$ the proper volume corresponding to a bin in $(z, \theta, \phi)$ and define

$$V_p = \bar{V}_p (1 + \mathcal{J}), \tag{44}$$

where $\bar{V}_p$ is the corresponding volume in an unperturbed Universe and[6]

$$\mathcal{J} = -\Phi - (1+z)\frac{d}{dz}\delta z_{G\to z} - 2\frac{1+z}{Hr}\delta z_{G\to z} - \delta z_{G\to z} - 2\kappa + \frac{1+z}{H}\frac{dH}{dz}\delta z_{G\to z} + 2\frac{\delta r}{r}, \tag{46}$$

gives the geometrical projection effects computed in [1]. Finally, we have

$$n_{obs}(z, \theta, \phi) = n_p(z, \Omega_K = 0)\left[1 - b_{\Omega_K}\Omega_K + \frac{\partial \log \bar{n}_p}{\partial \ln(1+z)}(\delta z_{G\to L} - \delta z_{G\to z}) + \mathcal{J}\right]. \tag{47}$$

Note that all the terms in the bracket are first order, i.e., they can be evaluated at $z$, $z_G$ or $t_G$ equivalently, since these agree at zeroth order.

### 4.1.3   Observed Overdensity & Averaging

The observed overdensity is the fractional difference between the overdensity in a certain direction and the angular average over the survey area

$$\delta_{obs}(z, \theta, \phi) = \frac{n_{obs}(z, \theta, \phi) - \bar{n}_{obs}(z)}{\bar{n}_{obs}(z)}. \tag{48}$$

When evaluating the observed mean number density we can use that all the terms in the bracket in eq. (47) vanish, when averaged over a sufficiently big survey area. Hence we obtain for the angular average

$$\bar{n}_{obs}(z) = \int_{\Omega_{survey}} \frac{\sin\theta d\theta d\phi}{\Omega_{survey}} n_{obs}(z, \theta, \phi) = n_p(z; \Omega_K = 0). \tag{49}$$

Now, the benefit of evaluating prefactor in eq. (47) at the observed redshift becomes obvious. Since the observed redshift is fixed, $n_p(z, \Omega_K = 0)$ agrees with the survey average and we have for the observed overdensity (we will ignore the additional effects on monopole and dipole, which are influenced by the contributions at the observer's position),

$$\delta_{obs}(z, \theta, \phi) = -b_{\Omega_K}\Omega_K + \frac{\partial \log \bar{n}_p}{\partial \ln(1+z)}(\delta z_{G\to L} - \delta z_{G\to z}) + \mathcal{J}. \tag{50}$$

The volume distortion is in principle observable and thus has to be gauge invariant by itself. The first term $-b_{\Omega_K}\Omega_K$ is the number of collapsed objects in the inertial frame and thus

---

[6]Our expression for $\mathcal{J}$ assumes that the survey is volume limited. If instead the survey is flux limited, we have to add the corrections due to the change in the apparent luminosity. In this case we have to replace $\mathcal{J}$ with

$$\mathcal{J} \quad \to \quad \mathcal{J} - 5p\,\delta\mathcal{D}_L. \tag{45}$$

See app. F for details.

totally independent of the choice of coordinates on the global manifold. The remaining redshift lapse is gauge invariant as we show in app. F. Together with the first term it forms another observable. With the above results the observed number density can be written as

$$n_{obs}(z, \theta, \phi) = \bar{n}_{obs}(z) \left[ 1 + \delta_{obs}(z, \theta, \phi) \right] . \tag{51}$$

We can also relate the expression in (47) to the overdensity in the global coordinates, $\delta_G(t_G, \vec{x}_G) = (n_p(t_G, \vec{x}_G) - \bar{n}_p(t_G))/\bar{n}_p(t_G)$, where the averaging is done over hypersurfaces of constant coordinate time. We obtain

$$n_{obs}(z, \theta, \phi) = \bar{n}_{obs}(z) \left[ 1 + \delta_G - \frac{\partial \log \bar{n}_p}{\partial \log(1+z)} \delta z_{G \to z} + \mathcal{J} \right] . \tag{52}$$

Note that in the case where the tracer has a number density that scales like $(1 + z)^3$ the combination $\delta_G - (\partial \log \bar{n}_p / \partial \log(1+z)) \, \delta z_{G \to z}$ becomes $\delta_G - 3\delta z_{G \to z}$ in agreement with [1].

In our formalism it was natural to define the bias directly in terms of the Laplacian of the $\zeta$ perturbation at the point of interest, which in turn is proportional to the curvature of the local FRW Universe. Because of the Friedmann equation, the curvature turns out to be proportional to the overdensity of the Universe at the source galaxy position, as shown next. This offers us a procedure to extract the bias from $N$-body simulations: run simulations with varying $\Omega_K$, and then take the derivative with respect to this parameter.

## 4.2 Comparison with Standard Newtonian Treatment of Bias

Our bias definition tells us that we should take the derivative of the number density with respect to the curvature of the local Universe. While our receipe is well defined in the full General Relativistic setup, it still should agree in the limit in which the long mode is well inside the horizon, so that the Newtonian approximation is valid for the long mode itself. However, in this case a naive look at the expression might make us think that the two procedures do not agree. Indeed, in the classical Newtonian treatment, the bias is defined as the derivative of the number density with respect to the local long-wavelength overdensity. In this section we will first relate the above bias definition to an overdensity and then consider the subhorizon limit.

The curvature energy density of the local Universe scales as $\Omega_K = \Omega_{K,0} H_0^2/(a_G H_G)^2$ and is thus fully specified by its value at redshift 0. The latter can be related to the matter density in (synchronous) comoving gauge as

$$\Omega_{K,0} = -\frac{K}{H_0^2} = \frac{2}{3} \frac{\nabla^2 \zeta}{H_0^2} = -\left(1 - \frac{f_0 H_0^2}{\dot{H}_0}\right)(1 - \Omega_{DE,0})\delta_{l,0}^{(com)} \tag{53}$$

$$= -\left(\Omega_{m,0} + \frac{2}{3} f_0\right) \delta_{l,0}^{(com)} \tag{54}$$

where the last two equalities are valid for a Universe with time varying dark energy and a $\Lambda$CDM Universe, respectively. Hence the bias term in eq. (47) can be written as

$$-b_{\Omega_K}(t)\Omega_K(t) = b_{\Omega_K}(t) \left(1 - \frac{f_0 H_0^2}{\dot{H}_0}\right) \frac{(1 - \Omega_{DE,0})H_0^2}{D(t)H(t)^2 a(t)^2} \delta_l^{(com)}(t) \equiv b(t)\delta_l^{(com)}(t) \tag{55}$$

18

We restored the time dependence of the long wavelength density perturbation, dividing by the linear growth factor $D(t)$. Our new bias $b_{\Omega_K}$ is related to the standard bias parameter by a time dependent but scale independent factor. From the above equation we can see that the density perturbation in the comoving gauge is equally suited, at an algebraical level, as an expansion parameter for the galaxy bias, but the justification of this statement relies simply on the proportionality of $b$ to $b_{\Omega_K}$. Further, the bias expressed in terms of $\Omega_K$ makes manifest its gauge-invariant physical origin and the fact that the biasing vanishes for modes longer than the Hubble scale.

Well inside the horizon $(k \gg aH)$ the velocity term in the relation between comoving and Newtonian gauge matter overdensity (see eq. (156) in app. F) becomes negligible and thus both density perturbations reduce to the Newtonian density perturbation $\delta_l^N \approx \delta_l^{com} \approx \delta_l$. Furthermore, inside the horizon the volume distortion as well as the lapse between the global, local and observed redshift are negligible. Thus eq. (50) reduces to

$$\delta_{obs}(z, \theta, \phi) = -b_{\Omega_K}(t)\Omega_K(t) = b(t)\delta_l(t) \; , \tag{56}$$

which is the standard relation between observed tracer overdensity and underlying matter overdensity in the Newtonian approximation.

Finally, we point out that another way to understand the connection between our bias $b_{\Omega_K}$ and the standard one is by referring to the peak background split method. There, in the Newtonian context, it is usually assumed that the presence of a long scale mode can be interpreted as a shift of $\delta_c$: $\delta_c \rightarrow \delta_c - \delta_l$, and after Taylor expansion we obtain the expression for the linear bias. In our context, the presence of a long mode is instead interpreted as a curvature of the background Universe, and therefore we have to rescale $\delta_c$ accordingly to $\delta_c(\Omega_K = 0) \rightarrow \delta_c(\Omega_K \neq 0)$ and then Taylor expand. In app. E, we show that indeed the two approaches are equivalent on short scales.

## 4.3  Bias in Presence of non-Gaussianities of the Local Kind

So far we have assumed that the only way a long-wavelength mode affects the local structure formation is through its dynamical effects: that is by changing the local geometry and by introducing curvature in the resulting local FRW Universe. If the initial conditions are Gaussian, this accounts for all the effects of the long mode on local processes: in the linear regime the statistical properties of the short wavelength modes are decoupled from long wavelength modes, and the non-linearities kick in only at late times on small scales, where all the effect of the long mode can be absorbed by a redefinition of the local expansion history. If the initial conditions are non-Gaussian, then the statistical properties of the initial short scale fluctuations are in general affected by the presence of a long mode and this has to be taken into account. The scales that become non-linear are very small compared to the horizon, and the scales that we are interested in are much larger than the non-linear scale. Thus, in order for the properties of the short scale fluctuations to be affected by the long mode, the non-Gaussian initial conditions need to be such that they correlate very long and very short modes.

In general the description of the statistical distribution of modes in the initial conditions requires knowledge of all the moments of their distribution. For special cases a limited set of parameters $\mathbf{p}$ is sufficient. For instance, if the initial conditions are Gaussian, they are fully quantified by their variance. If the parameters $\mathbf{p}$ depend on the long wavelength amplitude, then the proper number density of objects has an additional explicit dependence on the long wavelength amplitude. Thus we can generalize eq. (30) to:

$$n_p(\vec{x}_G, t_G; \zeta) = n_p\big(\vec{x}_L(\vec{x}_G, t_G), t_L(\vec{x}_G, t_G); \Omega_K, \mathbf{p}(\zeta)\big) \ . \tag{57}$$

These parameters $\mathbf{p}$ represent all the relevant information needed to describe the initial conditions on small scales. The abundance of objects of a given mass $M$ is mainly sensitive to the amplitude of fluctuations smoothed on a scale enclosing the mass, given in terms of the variance $\sigma_M$. There is also a weak dependence on the slope of the power spectrum at the scale $M$ and possibly on parameters describing deviations from a Gaussian distribution of the small scale modes, e.g. skewness. For definiteness we will consider only the dependence on $\sigma_M$.

The so-called local kind of non-Gaussianities [25] that can be produced in multifield inflationary models [26, 27] or in the new bouncing cosmology [28] provides an example where $\sigma_M$ depends explicitly on the long wavelength amplitude $\zeta$ [7]. In these models the initial conditions are such that the curvature perturbation is a non-linear function (local-in-space) of an auxiliary Gaussian random variable $\zeta_g$:

$$\zeta(\vec{x}_G) = \zeta_g(\vec{x}_G) - \frac{3}{5} f_{\mathrm{NL}}^{\mathrm{loc.}} \left( \zeta_g(\vec{x}_G)^2 - \langle \zeta_g^2 \rangle \right) \ . \tag{58}$$

If we decompose $\zeta$ into long and a short modes as we did before, we can see that the short mode takes the form

$$\zeta_s \simeq \left( 1 - \frac{6}{5} f_{\mathrm{NL}}^{\mathrm{loc.}} \zeta_{g,l} \right) \zeta_{g,s} \ , \tag{59}$$

where we have neglected a term in $\zeta_s^2$ which is irrelevant for our discussion of the bias. From this equation we see that the variance of the short scale power is modulated by the long mode. This implies that in the case of local non-Gaussianities there is an additional source of bias. If we set up the initial conditions for the simulation in the presence of non-Gaussianities of the local kind, the resulting proper number density of halos $n_p$ will depend on the long-mode not only through its explicit dependence on the curvature of the local Universe, but also through the dependence on the initial power spectrum of the modes [8]. Eq. (47) is generalized to

$$
\begin{aligned}
n_{obs}(z, \theta, \phi) &\simeq n_p(z; \Omega_K = 0, \bar{\mathbf{p}}) \times \\
&\quad \left[ 1 - b_{\Omega_K} \frac{2 \nabla_G^2 \zeta}{3 a^2 H^2} + \frac{1}{\bar{n}_p} \frac{\partial n_p}{\partial \sigma_M^2} \frac{\partial \sigma_M^2}{\partial \zeta} \zeta + \frac{\partial \log \bar{n}_p}{\partial \log(1+z)} \left( \delta z_{G \to L} - \delta z_{G \to z} \right) + \mathcal{J} \right] \\
&= \bar{n}_p(z) \left[ 1 - b_{\Omega_K} \Omega_K + b_\zeta \zeta + \frac{\partial \log \bar{n}_p}{\partial \log(1+z)} \left( \delta z_{G \to L} - \delta z_{G \to z} \right) + \mathcal{J} \right] , \tag{60}
\end{aligned}
$$

---

[7]The same is expected to be true for the new non-Gaussian shapes that have been found in the Effective Theory of Multifield Inflation that have support both in the equilateral and in the squeezed limit [4].

[8]Though we are now talking about non-Gaussian effects, notice that we are consistently treating the long mode at linear level.
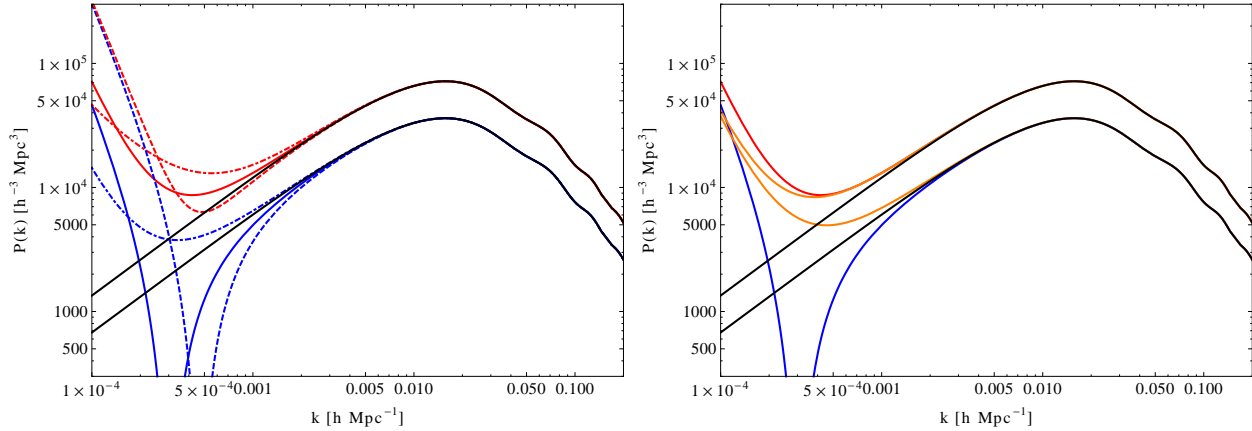
Figure 2: Observed galaxy power spectrum for $z = 1$, $b_{\Omega_K} = 1.5$ $(b = 2)$ and $\partial \log n_p / \partial \log(1 + z) = 3$. We choose the following cosmological parameters: $\Omega_m = 0.28$, $\sigma_8 = 0.84$, $H_0 = 0.70$. *Left panel:* We show the spectra parallel to the line of sight (red) and transverse to the line of sight (blue). The solid line is for Gaussian initial conditions, whereas dot-dashed is $f_{\mathrm{NL}}^{loc.} = +0.5$ and dashed is $f_{\mathrm{NL}}^{loc.} = -0.5$. The lower black line is just the power spectrum of density in comoving gauge, the upper is multiplied by the redshift space distortion factor $(1 + f/b)^2$ to give the power parallel to the line of sight. We see that the effects of non-Gaussianity and GR-effects on the power spectrum differ, because the latter depend also on the line of sight parameter $\mu$ through the peculiar velocity effects. *Right panel:* Same as left, but orange lines show non-Gaussian power spectrum without the GR-effects (just redshift space distortions).

where $\partial \sigma_M^2 / \partial \zeta = -12 f_{\mathrm{NL}}^{loc.}/5$ is independent of $M$ and $\bar{\mathbf{p}}$ describes the initial conditions in absence of long perturbations. We see that in presence of non-Gaussianities of the local kind the bias receives an additional contribution proportional to $\zeta$, while the standard Gaussian contribution is proportional to $\nabla^2 \zeta$. There is a relative scale dependence proportional to $k^2$ between the two. But this does not imply the very unphysical result that the bias blows up as $k \to 0$. It is rather the fact that the bias for large scales should be interpreted as a different bias: as the coefficient of proportionality between the local number density and $\zeta$ and $\nabla^2 \zeta$ [9]. The final expression for the observed overdensity in presence of local non-Gaussianities is thus

$$\delta_{obs}(z, \theta, \phi) = -b_{\Omega_K} \Omega_K + b_\zeta \zeta + \frac{\partial \log \bar{n}_p}{\partial \ln(1 + z)} \left( \delta z_{G \to L} - \delta z_{G \to z} \right) + \mathcal{J} \ . \tag{61}$$

The presence of $\Phi$ terms in the redshift lapse terms and the volume distortion term mimicks $f_{\mathrm{NL}}^{loc.}$ of order unity. But this should not bias any measurement of non-Gaussianity since the General Relativistic effects are calculable and can thus be removed from the measurement.

---

[9] Our conclusions about the bias as due to local non-Gaussianities are in general agreement with the ones of [12], though they differ in the way the results are derived and in parts of their interpretation. We stress that our derivation does not crucially rely on the assumption of spherical symmetry. It should allow for a straightforward generalization to the non-linear case where spherical symmetry can not be used.

As in the case of Gaussian initial conditions our formula can be applied directly to the results of $N$-body simulations, but for illustrative purposes we can also calculate the effect analytically assuming that the number density of collapsed objects is described by a universal mass function

$$n_p \propto f\left(\frac{\delta_c}{\sigma_M}\right) \; , \tag{62}$$

i.e., it is a function of the peak height, the ratio of collapse threshold and fluctuation amplitude. In this case the derivatives of $n_p$ with respect to $\sigma_M^2$ and with respect to the curvature $\nabla^2\zeta$ are related:

$$\begin{aligned}
\frac{\partial n_p}{\partial \zeta} &= \frac{\partial n_p}{\partial \sigma_M^2}\frac{\partial \sigma_M^2}{\partial \zeta} = -\frac{1}{2}\frac{\partial n_p}{\partial \delta_c}\frac{\delta_c}{\sigma_M^2}\frac{\partial \sigma_M^2}{\partial \zeta} \; , \\
\frac{\partial n_p}{\partial(-\Omega_K)} &= \frac{\partial n_p}{\partial \delta_c}\frac{\partial \delta_c}{\partial(-\Omega_K)} \; ,
\end{aligned} \tag{63}$$

which means that the Gaussian and the non-Gaussian bias are analytically related.

For general initial distributions, the additional contribution to the fluctuations in the proper number density arise from $\partial \mathbf{p}/\partial\zeta$. In the case of $\sigma_M$ considered above, this is nothing but the squeezed limit of the three point function, because this measures the coupling between small and large scale modes. Squeezed in this context refers to the fact that we are talking about a correlation between short and long wavelengths so two of the momenta in the relevant three point function are very large compared to the other one and thus the three momenta form a squeezed triangle. In the local model that we use as an illustration, the derivative

$$\frac{\partial \sigma_M^2}{\partial \zeta(k)} \tag{64}$$

is independent of both $k$ and of $M$ (we have explicitly pointed out that the derivative might be different as a function of the wavenumber of the long momenta). Relatively simple models can and have been constructed where this derivative depends on both $M$ and/or $k$ [29]. Even when this derivative is not constant our formulas remain valid.

As an illustration, in fig. 2 we show an example for the the observed galaxy power spectrum for $b_{\Omega_K} \approx 1.5$ ($b = 2$) at $z = 1$ assuming a volume limited survey. The plots show the power parallel and orthogonal to the line of sight for a sample with evolution slope of $\partial \log \bar{n}_p/\partial \log(1 + z) = 3$. We are adding local non-Gaussianity of $f_{\mathrm{NL}}^{loc.} = \pm 1$. The right panel shows that ignoring the GR-effects could lead to a fake detection of $f_{\mathrm{NL}} = \mathcal{O}(1)$, but this degeneracy is broken if modes both transverse and along the line of sight are considered. This is because GR-effects have a peculiar velocity contribution that has a $\mu$ dependence, where $\mu = \cos\theta$ and $\theta$ is the angle between the Fourier mode angle and the line of sight. For a related study on distinguishing GR effects from primordial non-Gaussianity see [30]. Note that the magnitude of the GR-effects depends on the redshift distribution of the sample. The non-Gaussian bias parameter $b_\zeta$ is calculated from the Gaussian bias using eq. (63). For the evaluation we are neglecting all the line of sight integrals (convergence, Shapiro-delay, integrated Sachs-Wolfe effect), which contribute power mainly to transverse modes. For the

details of the evaluation of the observed power spectrum, we refer the reader to eq. (174) in app. F.

It is also important to note that the relevant quantity is the change in the amplitude of fluctuations at a given physical scale $M$ not of course a comoving scale. In single field inflationary models this derivative goes to zero in the squeezed limit, when $k$ corresponds to a much larger scale than $M$. In fact it goes to zero as the square of $k$ just because the long wavelength mode affects the production of the short modes during inflation only through tidal type effects. In a sense it goes to zero in this way for reasons identical to the ones that lead to the $\nabla_G^2 \zeta$ dependence in the bias formulas. Thus, in single field inflationary models there is no modulation of the proper number density that scales with lower powers of $k$ than $\nabla_G^2 \zeta$ [10]. The reader familiar with the standard calculation of the single-field inflationary three point function might recall that in the squeezed limit they do not seem to vanish but that they satisfy a consistency condition where the shape of the three point function looks like that of a local model with an amplitude given by the tilt of the fluctuations usually called $(n_s - 1)$. But this dependence arises entirely from the fact that what is being calculated is a three point function in terms of comoving momenta. If expressed in terms of physical momenta, the $(n_s - 1)$ is exactly the amplitude required to make the relevant derivative vanish.

## 4.4 Observing Local-type non-Gaussianities in the Presence of GR Corrections

The salient fact about the local-type non-Gaussianities is that they induce a dependence of the proper number density of objects on the long wavelength modes that is much stronger than what the dynamical effects can produce, proportional to $\zeta$ rather than $\nabla_G^2 \zeta$. Unfortunately when we count objects in our Universe there are projection type effects that make the observed densities depend directly on $\zeta$ even if the proper density does not. The volume corresponding to a given observed range of angles and redshifts varies as a result of the long wavelength modes and results in the factor of $\mathcal{J}$ in eq. (47). Furthermore a given observed redshift corresponds to a different proper time in different directions resulting in the terms proportional to $\partial \log \bar{n}_p / \partial \log(1 + z)$ . Both of these terms lead to contributions proportional to $\zeta$, contributions that have the same form as that coming from the local-type of non-Gaussianities. Failing to correct for them would bias the results for $f_{\mathrm{NL}}^{\mathrm{loc.}}$ by a number of order one which depends on the details of the population of objects surveyed.

Of course the various terms have different dependences on the properties of the objects as they depend on different derivatives of $n_p^L$. The effects will also depend differently on redshift and furthermore, because the GR effects are projection effects induced by the intervening matter, it may be possible to distinguish them using observations of the distribution of matter at the intervening redshifts. It is beyond the scope of this paper to quantify the extent to which these different effects may be isolated in practice or what it is required of the observations to distinguish them.

---

[10]One can construct examples were there is an intermediate "squeezed regime" over which the scaling is different than $k^2$ but for sufficiently large ratio the scaling needs to be $k^2$ [31, 32, 33, 34, 35, 36].

It is clear however that the GR effects are just projection effects, so if we were able to construct observables that were directly sensitive to quantities in the local frame we could side track those difficulties. In this section we just want to point out that this is in principle possible. We will not address wether this can be done in practice given our current tools or wether this route is better than just trying to correct for the projection distortions in a realistic situation.

To be able to ignore the projection effects we would need to be able to measure the proper density of some object at a given proper time. Thus we would need a ruler that would allow us to measure distances independently of the observed angles and redshifts and we would need a clock that would allow us to compare regions of the Universe at the same proper time independently of the observed redshift. If we managed to find such local clocks and rulers the observed density should only depend on $\nabla^2_G \zeta$ in the absence of primordial non-Gaussianity. In fact there should only be a $\nabla^2_G \zeta$ dependence in any single field model of inflation.

There are many such rulers that one could imagine using. One option is to use the acoustic scale. This could be used for example by measuring the the number of objects in regions of a given size in units of the acoustic scale. The acoustic scale can be determined by measuring the correlation function of these or other objects. Another option is to measure the ratio of the densities of two tracers. Then the volume projection effects would cancel, in a sense we are using the density of one of the objects to define the ruler for the other.

We still need a clock to make sure that one is comparing the number densities at a fixed proper time rather than observed redshift. This difference is responsible for the terms proportional to $(\partial \log \bar{n}_p / \partial \log(1 + z))$ in eq. (47). This appears a bit more tricky but not a problem of principle. One needs to date the object observed independently of their redshift, something that happens automatically for tracers that appear only at a characteristic time in the history of the Universe. Examples of such things might one day be the first stars or perhaps quasars could be used as their abundance has a peak in redshift. In other words, the ratio of densities of tracers that come from a given proper time and could be identified without using the observed redshift would only depend of the long wavelength modes through the $\nabla^2_G \zeta$.

A similar construction for measuring the three point function in the squeezed limit could be accomplished using the CMB. The CMB comes already from a defined proper time, the recombination of hydrogen provides the clock. So one could use the dependence of the small scale power on large modes as a test of the squeezed three point function. One should use a local definition for fluctuations and power, meaning normalizing the fluctuations to the mean fluctuation level in the region of interest to eliminate the equivalent of the $\delta z_{G \to z}$ term in our equations for the densities of haloes. There is still the projection effect related to the mapping between angles and physical distances at recombination. This however can be avoided by comparing the amplitude of fluctuations at a fixed scale measured in units of the acoustic scale, thus at a fixed physical scale. For example the amplitude of the power spectrum at the $N$-th peak should only depend on the long modes through the $\nabla^2_G \zeta$ in the absence of primordial non-Gaussianities. One could also use the anisotropies in the small scale power to de-lens the CMB along the lines considered in [37, 38].

# Acknowledgments

# Appendix

# A    Fermi Coordinates from $\zeta$-gauge

Here we give the change of coordinates necessary to go from $\zeta$-gauge to the Fermi coordinates in the case of a spherically symmetric perturbation.

In $\zeta$-gauge the metric takes the form

$$ds^2 = -N^2 dt^2 + \delta_{ij} e^{2\zeta} a^2 \left( dx^i + N^i dt \right) \left( dx^j + N^j dt \right) , \tag{65}$$

where we have used the ADM parametrization. In this gauge time diffeomorphisms are fixed by requiring $T_i^0 = 0$. The lapse $N$ and shift $N^i$ are constrained variables, whose solutions in terms of $\zeta$ are [43]

$$N = 1 + \frac{\dot{\zeta}}{H} , \qquad N_i = -\frac{\nabla_{G,i}\zeta}{H} - \frac{\dot{H}}{H^2} \frac{a^2}{c_s^2} \frac{\nabla_{G,i}}{\nabla_G^2} \dot{\zeta} . \tag{66}$$

The equation of motion for $\zeta$ reads [43]

$$\frac{1}{a^3} \partial_t \left( \frac{a^3}{c_s^2} \frac{\dot{H}}{H^2} \dot{\zeta} \right) + \frac{\dot{H}}{H^2} \frac{\nabla_G^2 \zeta}{a^2} = 0 . \tag{67}$$

Outside the sound horizon and assuming $c_s$ constant, we can simplify it to

$$\dot{\zeta} = \frac{\dot{H}}{H^2} \frac{c_s^2}{\left( \partial_t \left( \frac{\dot{H}}{H^2} \right) + 3\frac{\dot{H}}{H} \right)} \frac{\nabla_G^2 \zeta}{a^2} . \tag{68}$$

Plugging back in (66), we can simplify the expression for $N$ and $N_i$ to be

$$N \simeq 1 , \qquad N_i = - \left( \frac{1}{H} + \frac{\dot{H}}{H^2} \frac{1}{3H - \partial_t \left( \dot{H}/H^2 \right)} \right) \nabla_{G,i}\zeta . \tag{69}$$

At this point we proceed as in the main text. We Taylor expand $\zeta$ around the origin assuming spherical symmetry, we make an ansatz for the change of coordinates, and we impose the resulting metric to be in the Fermi form. After some straightforward algebra, we obtain for the change of coordinates

$$
\begin{aligned}
t_G &= t_L - \frac{1}{2}\left[H(t_L) + \frac{\zeta_{,r_G r_G}}{a^2}\left(1 - \frac{H\dot{H}}{-3H^4 - 2\dot{H}^2 + H\ddot{H}}\right)\right]r_L^2 \, , \\
x_G^i &= \frac{x_L^i}{a(t_L)}\left[1 + \frac{H(t_L)^2}{4}r_L^2\right]\left(1 - \zeta(\vec{0})\right) \, ,
\end{aligned}
\tag{70}
$$

and for the metric

$$
ds^2 = \tag{71}
$$

$$
\begin{aligned}
-&\left\{1 - \left[\dot{H}(t_L) + H(t_L)^2 - \frac{\zeta_{,r_G r_G}}{a^2}\frac{1}{H^2\left(3H^4 + 2\dot{H}^2 - H\ddot{H}\right)^2}\left(9H^8\dot{H} + 9H^6\dot{H}^2 + 4\dot{H}^5 - 3H^7\ddot{H}\right.\right.\right. \\
&\left.\left.\left. -6H^5\dot{H}\ddot{H} + 2H^3\dot{H}^2\ddot{H} - 4H\dot{H}^3\ddot{H} + H^2\dot{H}\left(-2\dot{H}^3 + \ddot{H}^2\right) + H^4\left(12\dot{H}^3 + \ddot{H}^2 - \dot{H}\ddot{H}\right)\right)\right]r_L^2\right\}dt_L^2 \\
&+\left\{1 - \left(\frac{H(t_L)^2}{2} - \frac{\zeta_{,r_G r_G}}{a(t_L)^2}\cdot\frac{H^2\dot{H}}{-3H^4 - 2\dot{H}^2 + H\ddot{H}}\right)r_L^2\right\}d\vec{x}_L^2 \, .
\end{aligned}
$$

As expected, this metric has the same form as the Fermi patch of a closed FRW Universe with

$$
H_L(t_L) = H(t_L) + \frac{\zeta_{,r_G r_G}}{a(t_L)^2}\left(\frac{1}{H} + \frac{H\dot{H}}{3H^4 + 2\dot{H}^2 - H\ddot{H}}\right) \, , \tag{72}
$$

$$
K_L = -\frac{2}{3}\nabla_G^2\zeta(\vec{0}, t_G) \, ,
$$

in agreement with what found in the Newtonian-gauge case.

# B  A Geometric Derivation of the Fermi Coordinates

In this section we will describe how the Fermi coordinates can be constructed from a geometric point of view.

The starting point for the derivation will be a free falling observer moving along a timelike geodesic $h(\gamma)$ in the background Universe (fig. 3). His coordinate axes are described by an orthonormal set $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ which is parallely transported along $h$. Thus if $\vec{e}_0$ is tangent to the geodesic $h(\gamma)$ at its origin it will remain so for all values of the affine parameter $\gamma$. Without loss of generality we can assume $\vec{e}_0$ to be timelike and the $\vec{e}_i$, $i = 1, 2, 3$ to be spacelike, and the geodesic to be the origin of the global coordinate frame $x_G^i = 0$.

Now we consider a point $P = h(\gamma_0)$ on this geodesic. Our goal is to describe the spacetime in a neighborhood $\mathcal{U}$ of $P$ starting from the global metric at $P$. Any point $Q$ in the vicinity
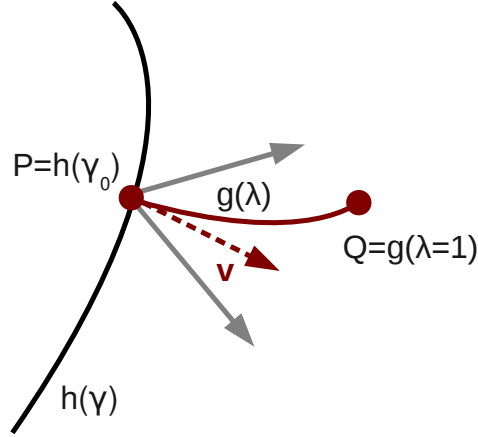
Figure 3: *Geometrical Construction of the Fermi Coordinates.*

of $P$ can be connected to $P$ with a geodesic $g(\lambda)$ that is perpendicular to the tangent vector of $h$ at $P$, i.e., its tangent vector $\vec{v}$ at $P$ is a linear combination of the $\vec{e}_i$. The coefficients of this linear combination are the Fermi coordinates and the time component of the Fermi coordinates is chosen to be the proper time of the observer moving along $h$. The point $Q$ can thus be fully described by the proper time $\tau$ of the observer at $P$, the direction cosines $x^i$ and the length of the geodesic $\lambda$ joining $Q$ with $P$. For simplicity we will normalize the direction cosines such that the point $Q$ corresponds to $\lambda = 1$. This prescription is the natural extension of the flat space polar coordinates to curved space. The observer points in a certain direction defined by the direction cosines $x_L^i$ and then follows the geodesic defined by the direction.

The initial conditions for the geodesic connecting $P$ and $Q$ can thus be summarized as

$$x^i(\lambda = 0) = 0 \ , \qquad\qquad \tau(\lambda = 0) = t_L \ , \tag{73}$$

$$\left.\frac{dt_G}{d\lambda}\right|_{\lambda=0} = 0 \ , \qquad\qquad \left.\frac{dx_G^i}{d\lambda}\right|_{\lambda=0} = x_L^i \vec{e}_i(t_0) \ .$$

The point $Q \in \mathcal{U}$ with Fermi coordinates $x_L^\mu$ is then found by propagating along $g(\lambda)$ until $\lambda = 1$. We now have to find the mapping between arbitrary coordinates $x^\mu$ and the Fermi coordinates defining the geodesic $g(\lambda)$. This can be done by solving the geodesic equation for $g(\lambda)$

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \ , \tag{74}$$

perturbatively using the power law ansatz

$$x^\mu(\lambda) = \alpha_0^\mu + \alpha_1^\mu \lambda + \alpha_2^\mu \lambda^2 + \alpha_3^\mu \lambda^3 + \dots . \tag{75}$$

The validity of this series is clearly limited as is the validity of the Fermi coordinates themselves, which is obviously related to the curvature of the spacetime. The four vector formu-

27

lation for the initial conditions stated above is

$$\alpha_0^\mu = (t_0, 0, 0, 0), \tag{76}$$

$$\alpha_1^\mu = \frac{dx^\mu}{d\lambda}\bigg|_{\lambda=0} = x_L^i [\vec{e}_i]^\mu.$$

where $t_0$ is the coordinate time corresponding to $\gamma_0$. The coefficients of the second and third order terms in the Taylor series follow straightforwardly from the geodesic equation evaluated at $P$

$$\alpha_2^\mu = \frac{1}{2!} \frac{d^2 x^\mu}{d\lambda^2}\bigg|_{\lambda=0} = -\frac{1}{2} \Gamma^\mu_{\gamma\nu} \alpha_1^\gamma \alpha_1^\nu, \tag{77}$$

$$\alpha_3^\mu = \frac{1}{6!} \frac{d^3 x^\mu}{d\lambda^3}\bigg|_{\lambda=0} = -\frac{1}{6} \left( \frac{\partial \Gamma^\mu_{\gamma\nu}}{\partial x^\kappa} \alpha_1^\gamma \alpha_1^\nu \alpha_1^\kappa + 4 \Gamma^\mu_{\gamma\nu} \alpha_1^\gamma \alpha_2^\nu \right),$$

where we already simplified using the initial conditions. In the following two subsections we will describe the mapping for two specific cases: perturbed and unperturbed FRW Universes.

## B.1 FRW

We will now follow the above procedure for the homogeneous Friedmann-Robertson-Walker metric

$$ds^2 = -dt_G^2 + a(t_G)^2 \frac{d\vec{x}_G^2}{\left[1 + \frac{1}{4} K \vec{x}_G^2\right]}, \tag{78}$$

The vierbein associated to a comoving geodesic is

$$[\vec{e}_0]^\mu = (1, 0, 0, 0) , \qquad\qquad [\vec{e}_1]^\mu = a^{-1}(0, 1, 0, 0) , \tag{79}$$

$$[\vec{e}_2]^\mu = a^{-1}(0, 0, 1, 0) , \qquad\qquad [\vec{e}_3]^\mu = a^{-1}(0, 0, 0, 1) .$$

The linear coefficients in the geodesic expansion read

$$\alpha_1^\mu = a(t_0)^{-1}(0, x_L, y_L, z_L) . \tag{80}$$

Hence, the first order spatial separation is $x_G^i \simeq x_L^i / a(t_0)$ and thus $x_L^i$ is nothing but the physical separation of $Q$ from $P$. Up to third order in the affine parameter we obtain

$$t_G = t_L - \frac{H \vec{x}_L^2}{2} , \tag{81}$$

$$x_G^i = \frac{x_L^i}{a(t_L)} \left( 1 + \frac{H^2 \vec{x}_L^2}{3} \right) .$$

This leads to the following metric in Fermi Normal coordinates

$$ds^2 = -\left[1 - \left(\dot{H}(t_L) + H^2(t_L)\right) \vec{x}_L^2\right] dt_L^2$$

$$+ \left[\delta_{ij} - \left(H^2(t_L) + \frac{K}{a^2}\right) \frac{\vec{x}_L^2 \delta_{ij} - x_L^i x_L^j}{3}\right] dx_L^i dx_L^j \tag{82}$$

28

The above metric has non-zero off-diagonal contributions. The general transformation to remove off diagonal terms can be derived considering the metric in the old coordinates $\tilde{x}$

$$ds^2 = \tilde{A}\,\delta_{ij}\,d\tilde{x}^i\,d\tilde{x}^j + \tilde{B}\,\tilde{x}_i\tilde{x}_j\,d\tilde{x}^i\,d\tilde{x}^j \,, \tag{83}$$

and new coordinates $x(\tilde{x})$

$$ds^2 = A\,\delta_{ij}\,dx^i\,dx^j \,. \tag{84}$$

Using the ansatz $\tilde{x}^i = x^i(1 + \gamma x^2)$ we obtain the condition valid at second order in $x$:

$$\gamma = -\frac{\tilde{B}}{4\tilde{A}} \tag{85}$$

$$A = \tilde{A}(1 + 2\gamma x^2) \,.$$

For the FRW case we have $\gamma = -H^2/12 - K/(12a^2)$ and the time component up to second order is unaffected

$$t_G = t_L - \frac{H(t_L)}{2}\vec{x}_L^2 \,, \tag{86}$$

$$x_G^i = \frac{x_L^i}{a(t_L)}\left(1 + \frac{H(t_L)^2}{4}\vec{x}_L^2\right) \,,$$

finally leading to the following metric

$$ds^2 = -\left[1 - \left(\dot{H}(t_L) + H(t_L)^2\right)\vec{x}_L^2\right]dt_L^2 + \left[1 - \left(H(t_L)^2 + \frac{K}{a(t_L)^2}\right)\frac{\vec{x}_L^2}{2}\right]d\vec{x}_L^2 \,, \tag{87}$$

which has the desired form.

## B.2   Perturbed FRW

Let us now consider a perturbed FRW Universe in Newtonian gauge

$$ds^2 = -\left(1 + 2\Phi(t)\right)dt^2 + a^2(t)\left(1 - 2\Psi(t)\right)d\vec{x}^2 \,. \tag{88}$$

We assume vanishing anisotropic stress leading to $\Phi = \Psi$. The vierbein associated to the coordinate frame is

$$[\vec{e}_0]^\mu = (1 - \Phi, 0, 0, 0) \,, \qquad\qquad [\vec{e}_1]^\mu = a^{-1}(0, 1 + \Phi, 0, 0) \,, \tag{89}$$

$$[\vec{e}_2]^\mu = a^{-1}(0, 0, 1 + \Phi, 0) \,, \qquad\qquad [\vec{e}_3]^\mu = a^{-1}(0, 0, 0, 1 + \Phi) \,.$$

We can now for simplicity expand the potentials around $P$

$$\Phi(\vec{x}, t) = \Phi(\vec{0}, t) + \frac{1}{2}\frac{\partial^2\Phi}{\partial r_G^2}\bigg|_0 r_G^2 = \Phi(\vec{0}, t) + \frac{1}{2}\Phi(\vec{0}, t)_{,r_G r_G}r_G^2 \,, \tag{90}$$

where we asumed spherical symmetry [11] leading to

$$ds^2 = -\left(1 + 2\Phi(\vec{0}, t) + \Phi(\vec{0}, t)_{,r_G r_G} r_G^2\right) dt_G^2 + a^2(t)\left(1 - 2\Phi(\vec{0}, t) - \Phi(\vec{0}, t)_{,r_G r_G} r_G^2\right) d\vec{x}_G^2, \quad . \tag{91}$$

There is no linear term in this expansion, because we require the potential to be differentiable at $r = 0$. Let us proceed to find the Fermi coordinates. As noted above, the Fermi time is the proper time of the observer following the central geodesic $h$

$$t_L = \int_0^t \sqrt{-g_{00}}\, dt' = t + \int_0^t \Phi(\vec{0}, t')\, dt' \ . \tag{92}$$

The coordinate time at $P$ thus is $t_0 = t_L - \int_0^{t_L} \Phi(\vec{0}, t')\, dt'$, where in the integral boundary $t_L = t_0$ at leading order in $\Phi$. This leads to the following expansion factors

$$\alpha_0^\mu = (t_0, 0, 0, 0) \ , \tag{93}$$

$$\alpha_1^\mu = \frac{1 + \Phi(\vec{0}, t_0)}{a(t_0)}\left(0, x_L, y_L, z_L\right) \ .$$

At this point, simple algebra as shown in the former section leads to the same relationship among the coordinates as in (17) and to the same Fermi metric as in (19).

## B.3   Local Expansion Factor

With the aim of giving very specific recipe for running simulations given a certain long wavelength fluctuation, we provide some more specific relations. Some expressions can be simplified by noticing that the potential in Newtonian gauge and the density perturbation in the comoving gauge are related by (see Appendix F)

$$\nabla^2 \Phi(\vec{x}, t) = 4\pi G a^2 \bar{\rho}\, \delta_l^{(com)}(\vec{x}, t) \ . \tag{94}$$

Note that this equation is exact, even on horizon scales. We define the growth factor in comoving synchronous gauge as $\delta_l^{(com)}(t) = D(t)\delta_{l,0}^{(com)}$, which is normalised to unity at present time. We also define the logarithmic growth factor $f(a) = d\ln D/d\ln a$. From eq. (94), we define the growth factor of the Newtonian potential $D$ as follows $\Phi(t) = D(t)\Phi_0/a(t)$, where $\Phi_0$ is the present day value. From the linear growth and eq. (94) it follows

$$\Phi(\vec{0}, t)_{,t r_G r_G} = H\Phi(\vec{0}, t)_{,r_G r_G}(f - 1) \ . \tag{95}$$

Using the constancy of $\zeta$ we have shown that the value of the metric perturbation at the origin is irrelevant for the local expansion. Thus it only remains to derive the rescaling of the

---

[11]Note that

$$\nabla^2 \Phi = \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial r^2} = 3\frac{\partial^2 \Phi}{\partial r^2}$$

where the last equality is true for a power law in $r$ assuming no linear dependence in $r$.

expansion factor corresponding to the effective local Hubble rate. Starting from (20), which, by defining $H_L(t) = H_G(t) + \delta H(t)$, gives

$$\delta H = \frac{1}{a_G^2(t)} \frac{H_G(t)}{\dot{H}_G(t)} \left( \Phi(\vec{0}, t) + \frac{\Phi(\vec{0}, t)_{,t}}{H} \right)_{,r_G r_G} , \tag{96}$$

we can find the corresponding rescaling for the expansion factor using the ansatz

$$a_L(t) = a_G(t) \left( 1 + \delta a(t)_{rel} \right) , \tag{97}$$

where $\delta a_{rel}$ has to satisfy the following:

$$\dot{\delta a}_{rel}(t) = \frac{1}{3 a_G^2(t) H_G(t)} \nabla_G^2 \left( \Phi(\vec{0}, t) + \zeta(\vec{0}, t) \right) , \tag{98}$$

$$\Rightarrow \quad \delta a_{rel}(t) = \int_0^t dt' \frac{1}{3 a_G^2(t') H_G(t')} \nabla_G^2 \left( \Phi(\vec{0}, t') + \zeta(\vec{0}, t') \right) ,$$

where we have chosen the constant so that the two scale factors agree at early times. $\delta a(t)_{rel}$ can be numerically integrated from the transfer functions for any given cosmology.

Finally, the Friedmann equations in the Fermi frame read as

$$H_L^2 = \frac{8\pi G}{3} \bar{\rho}_L + \frac{8\pi G}{3} \bar{\rho}_{DE} - \frac{K}{a_L^2} , \tag{99}$$

and

$$\frac{\ddot{a}_L}{a_L} = -\frac{4\pi G}{3} \bar{\rho}_L + \frac{8\pi G}{3} \bar{\rho}_{DE} , \tag{100}$$

where $\bar{\rho}_L$ is the local mean matter density. From the rescaling between the global and local Hubble rate we can derive the rescaling of the local mean density

$$\frac{H_L(t)^2 + \frac{K}{a(t)^2} - \frac{8\pi G}{3} \bar{\rho}_{DE}(t)}{H_G(t)^2 - \frac{8\pi G}{3} \bar{\rho}_{DE}(t)} = \frac{\bar{\rho}_L(t)}{\bar{\rho}_G(t)} , \tag{101}$$

leading to

$$\bar{\rho}_L(t) = \bar{\rho}_G(t) + \frac{3\Phi(\vec{0}, t)_{,r_G r_G}}{4\pi G a^2(t)} = \bar{\rho}_G(t) \left( 1 + \delta_l^{(com)}(t) \right) . \tag{102}$$

This relation can be intuitively understood in the Newtonian context: the long wavelength density just rescales the local mean density. This relationship gets upgraded to the relativistic setup by using the comoving gauge overdensity.

For definiteness we give also the closed form expressions for the expansion and Hubble rate in a $\Lambda$CDM background

$$H_L(t) = H_G(t) \left( 1 - \frac{f(t)\Phi(\vec{0}, t)_{,r_G r_G}}{4\pi G a^2(t) \bar{\rho}_G(t)} \right) = H_G(t) \left( 1 - \frac{1}{3} f(t) \delta_l^{(com)}(t) \right) , \tag{103}$$

$$a_L(t) = a_G(t) \left( 1 - \frac{\Phi(\vec{0}, t)_{,r_G r_G}}{4\pi G a^2(t) \bar{\rho}_G(t)} \right) = a_G(t) \left( 1 - \frac{1}{3} \delta_l^{(com)}(t) \right) .$$

31

## B.4 Local Density Parameters

The time evolution of the local patch is determined by the local Friedmann eqns. (99) and (100), which are parametrized by the effective local density parameters. We will now provide the explicit mapping from the global to the local cosmological parameters that are needed for simulations. We specialize to $\Lambda$CDM for simplicity, though, as we stressed, our approach applies also to clustering dark energy. From the relationship between $H_G$ and $H_L$ and from the definition of of $K$ given in (20), we have

$$\Omega_{K,L}(t_L) = -\frac{K}{a_L(t_L)^2 H_L(t_L)^2} = \frac{2}{3}\frac{1}{a_L(t_L)^2 H_L(t_L)^2}\nabla_G^2\zeta(\vec{x}_G) \,, \tag{104}$$

$$\Omega_{\Lambda,L}(t_L) = \frac{\Lambda}{3H_L^2} \,, \qquad \Omega_{m,L}(t_L) = 1 - \Omega_K(t_L) - \Omega_\Lambda(t_L) \,.$$

It is convenient to normalize the cosmological parameters at $a = 1$ (for us $a_L = 1$), which leads to

$$\Omega_{K,L,0} = -\frac{K}{H_{L,0}^2} = \frac{2}{3}\frac{1}{H_{L,0}^2}\nabla_G^2\zeta(\vec{x}_G) \,, \tag{105}$$

$$\Omega_{\Lambda,L,0} = \frac{\Lambda}{3H_{L,0}^2} \,, \qquad \Omega_{m,L,0} = 1 - \Omega_{K,0} - \Omega_{\Lambda,0} \,,$$

where the subscript $_0$ stays for evaluating the quantity when $a_L = 1$. In order to be able to use the above formulas, we simply need to find the time $t_L$ at which $a_L = 1$. This can be found by solving eq. (97) with $a_L = 1$ to identify $t_{L,0}$. From there, by plugging into (20) we get $H_{L,0}$. The former expressions can be further simplified to give

$$\Omega_{m,L,0} = \frac{8\pi G\bar{\rho}_L}{3H_{L,0}^2} = \Omega_{m,0}\left[1 - \frac{2}{3}\frac{\nabla_G^2\zeta(\vec{0})}{H_{G,0}^2}\right] = \Omega_{m,0}\left[1 + \left(\Omega_{m,0} + \frac{2}{3}f_0\right)\delta_{l,0}^{(com)}\right] \tag{106}$$

$$\Omega_{K,L,0} = -\frac{K}{H_{L,0}^2} = \frac{2}{3}\frac{\nabla_G^2\zeta(\vec{0})}{H_{G,0}^2} = -\left(\Omega_{m,0} + \frac{2}{3}f_0\right)\delta_{l,0}^{(com)}$$

$$\Omega_{\Lambda,L,0} = \frac{\Lambda}{3H_{L,0}^2} = (1 - \Omega_{m,0})\left[1 - \frac{2}{3}\frac{\nabla_G^2\zeta(\vec{0})}{H_{G,0}^2}\right] = (1 - \Omega_{m,0})\left[1 + \left(\Omega_{m,0} + \frac{2}{3}f_0\right)\delta_{l,0}^{(com)}\right] \,.$$

The local Hubble rate at $a_L = 1$ is given by

$$H_{L,0} = H_{G,0}\left(1 + \frac{1}{3}\frac{\nabla_G^2\zeta(\vec{0})}{H_{G,0}^2}\right) = H_{G,0}\left[1 - \frac{1}{2}\left(\Omega_{m,0} + \frac{2}{3}f_0\right)\delta_{l,0}^{(com)}\right] \,. \tag{107}$$

As an example we consider the WMAP5 Flat $\Lambda$CDM cosmology with matter density parameter $\Omega_{m,0} = 0.28$ and Hubble constant $H_{G,0} = 70\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$. For a long wavelength amplitude of $\delta_{l,0}^{(com)} = 0.1$ corresponding to $\nabla_G^2\zeta/H_{G,0}^2 = 0.89$ we obtain

$$\Omega_{m,L,0} = 0.30 \,, \qquad \Omega_{K,L,0} = -0.06 \,, \qquad \Omega_{\Lambda,L,0} = 0.76 \qquad h_{L,0} = 0.68 \,, \tag{108}$$
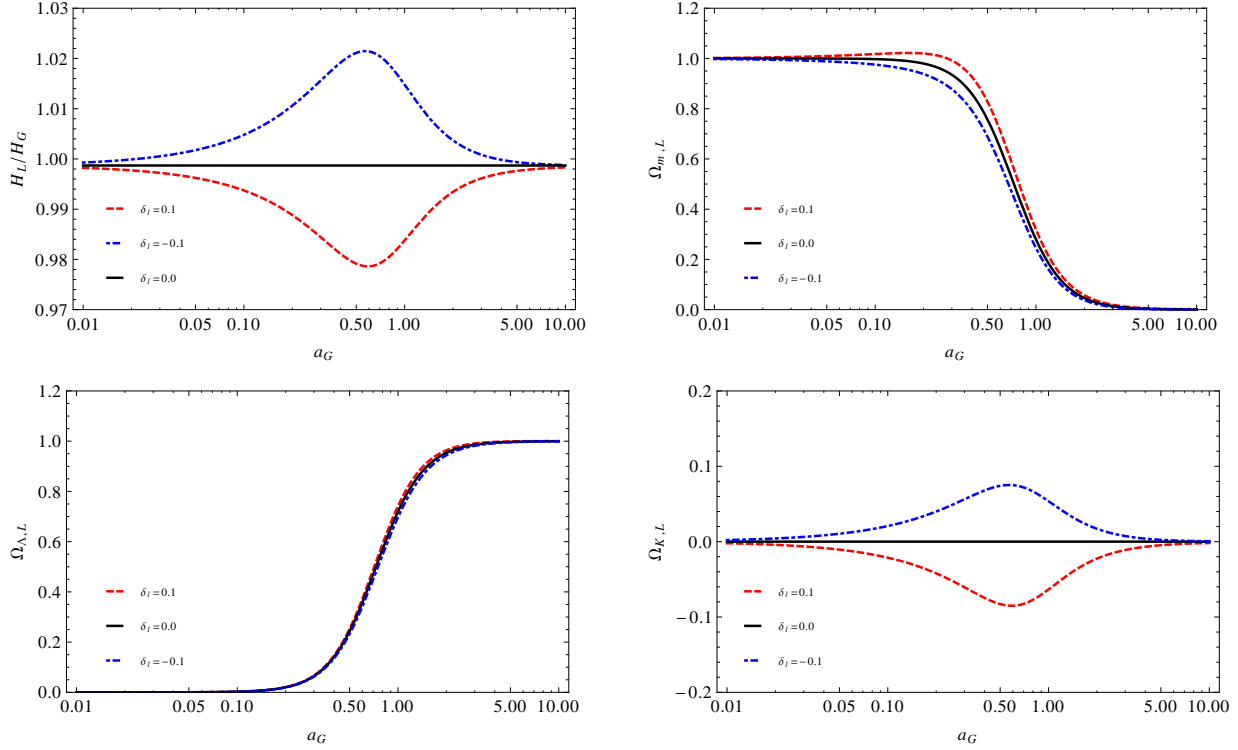
Figure 4: *Time dependence of the local expansion history as a function of the global expansion factor. In all panels the solid black line represents the flat background model, whereas the red dashed and blue dash-dotted lines represent an over- or underdense region.* Top left: *Ratio of the local and global Hubble rate.* Top right: *Local matter density parameter.* Bottom left: *Local cosmological constant density parameter.* Bottom right: *Local curvature density parameter.*

where we wrote the Hubble constant in terms of $h_L$ as $H_{L,0} = 100 h_L \, \mathrm{km \, s^{-1} \, Mpc^{-1}}$.

In fig. 4 we show the time dependence of the effective local expansion history. At early times the curvature is negligible and the effective local Universe approaches the flat background Universe. At late times, the cosmological constant dominates and thus the contribution of matter and curvature to the energy budget becomes irrelevant.

# C  Fermi Coordinates for Plane-Wave Perturbed FRW Universe

Let us assume we start in Newtonian gauge with $\Phi(\vec{x}_G, t_g)$ being a plane wave with wave-number that we can take without loss of generality in the $x$-direction

$$\Phi(\vec{x}_G, t_G) = \Phi_0 \, e^{ikx_G} \ . \tag{109}$$

We can find the Fermi coordinates around the origin by working as in the main text and assume a change of coordinates valid at cubic order in the spatial distance of the form:

$$t_G = t_L - \frac{1}{2}H(t_L)r_L^2 - \int_0^{t_L}\Phi(\vec{0},t')dt' + g_1(t_L)r_L^2 + \tag{110}$$

$$+g_{a,1;j}(t_L)x_L^j + g_{a,2;j,k}(t_L)x_L^j x_L^k \ ,$$

$$x_G^i = \frac{x_L^i}{a(t_L)}\left(1 + \frac{1}{4}H(t_L)^2 r_L^2 + f_1(t_L) + f_2(t_L)r_L^2\right)$$

$$+f_{a,tr}^i(t_L) + f_{a,0;j}^i(t_L)x_L^j + f_{a,1;j,k}^i(t_L)x_L^j x_L^k + f_{a,2;j,k,l}^i(t_L)x_L^j x_L^k x_L^l \ .$$

This represents the most general change of coordinates around $\vec{x}_G = \vec{0}$ at cubic order in the distance from the origin, and it is a straightforward generalization of (12). The subscript $a$ represents the fact that those functions are zero in the limit of isotropic perturbations. By imposing the metric in the new local coordinates to be of the Fermi form, we can proceed and identify the unknown functions. We skip the details associated to straightforward algebra, and just quote the final result. In order to limit the size of the expressions, we simply quote the simple expressions that are obtained after we restrict to the case of constant $\zeta$.

Under the following change of coordinates

$$t_G = t_L - \int_0^{t_L}\Phi(\vec{0},t_L)dt' - \frac{1}{2}H(t_L)\left(1 - \Phi(\vec{0},t_L)\right)r_L^2$$

$$+\frac{x_L^1}{a(t_L)H(t_L)}\nabla_{G,1}\left[\Phi(\vec{x}_G,t_L) + \zeta(\vec{x}_G,t_L)\right]|_{\vec{x}_G=0} \ ,$$

$$x_G^i = \frac{x_L^i}{a(t_L)}\left\{\left[1 + \frac{H(t_L)^2}{4}r_L^2\right]\left(1 - \zeta(\vec{0},t_L)\right) - \frac{x_L^1}{a(t_L)}\nabla_{G,1}\zeta(\vec{x}_G,t_L)\right\} \tag{111}$$

$$+\delta_{i,1}\left[\nabla_{G,1}\int_0^{t_L}dt'\frac{\Phi(\vec{x}_G,t_L') + \zeta(\vec{x}_G,t_L')}{H(t_L')a(t_L')^2}\bigg|_{\vec{x}_G=0} - \frac{1}{2}\frac{r_L^2}{a(t_L)^2}\nabla_{G,1}\Phi(\vec{x}_G,t_L)\right.$$

$$\left.+\frac{(x_L^1)^3}{6a(t_L)^3}\nabla_{G,11}^2\Phi(\vec{x}_G,t_L)|_{\vec{x}_G=0}\right] \ ,$$

we obtain the following metric components:

$$g_{00} = -1 + \left(H(t_L)^2 + \dot{H}(t_L)\right)r_L^2 - \frac{(x_L^1)^2}{a(t_L)^2}\nabla_{G,11}^2\Phi(\vec{x}_G,t_L)|_{\vec{x}_G=0} \ , \tag{112}$$

$$g_{0i} = \frac{H(t_L)}{4a(t_L)}\left(2x_L^i x_L^1 + \delta_{i1}r_L^2\right)\nabla_{G,1}\left[\zeta(\vec{x}_G,t_L) + 3\Phi(\vec{x}_G,t_L)\right]|_{\vec{x}_G} \ ,$$

$$g_{ij} = \delta_{ij}\left(1 - \frac{r_L^2}{2}H(t_L)^2 - \frac{(x_L^1)^2}{a(t_L)^2}\nabla_{G,11}\Phi(\vec{x}_G,t_L)|_{\vec{x}_G}\right) + \delta_{i1}\delta_{j1}\frac{(x_L^1)^2}{a(t_L)^2}\nabla_{G,11}^2\Phi(\vec{x}_G,t_L)|_{\vec{x}_G} \ ,$$

We see that in the anisotropic case, the metric has non-vanishing $0i$ components at order $r_L^2$. This form of the metric is important if we are interested in evaluating the bias for a non-scalar quantity, for which case the problem can not be reduced to the spherically symmetric case.

34

# D    Growth in Presence of a Long Mode

In the main text, we derived a change of coordinates that is valid in a small region around a given time-like geodesic and that allowed us to describe the effect of a long wavelength fluctuation effectively as a local closed FRW Universe. This change of coordinates is valid at linear order in the long mode and at any order in the short wavelength perturbations. In the main text we focus on collapsed objects, as our main interest is extracting information about halo bias. Therefore we follow the short scale power well into the non-linear regime. On the other hand, the mapping can also be used to analytically examine the coupling of linear short wavelength modes to long wavelength modes while the short modes are still in the quasi-linear regime. In this regime, we are now going to explicitly compare results derived in our formalism to the ones obtained in standard perturbation theory. Since standard perturbation theory is performed in the Newtonian limit ($k/aH \gg 1$), we will adopt the simplifying assumption that the long mode is sufficiently far inside the horizon that the Newtonian approximation holds also for the long mode. In this limit we can for example neglect $\Phi \ll \nabla^2\Phi/(a^2H^2)$ [12]. We will also restrict ourselves to the Einstein de-Sitter (EDS) Universe for simplicity.

As shown in the main text, the effect of a long wavelength mode on the local dynamics can be ascribed to a non-vanishing spatial curvature $K$ in a fictious closed global FRW Universe. The curvature parameter $\Omega_K$ and the curvature $K$ are related by

$$\Omega_K = -\frac{K}{a^2H^2}. \tag{113}$$

Let us begin to investigate the growth of short scale fluctuations in a closed FRW Universe. The linear growth equation for the short wavelength matter density perturbations reads as

$$\ddot{\delta}_s + 2H\dot{\delta}_s - 4\pi G\bar{\rho}\delta_s = 0 . \tag{114}$$

This equation is solved by the linearly growing modes $\delta_s(t) = D(t)\delta_{s,0}$:

$$D(t) = \frac{5}{2}\Omega_m H_0^2 H(t) \int_0^{a(t)} \frac{d\tilde{a}}{[\tilde{a}H(\tilde{a})]^3} , \tag{115}$$

which in EDS simplifies to $D(t) = a(t)$, where we use the subscript $_0$ to indicate present time and where we have normalized $a_0 = 1$. We will now look at the relation between the linear growth in a globally flat Universe and the effective local curved Universe. Using $\dot{H}(t) = -3H^2(t)/2$ in EDS, we obtain for the local effective curvature in terms of the long wavelength density perturbation

$$K = 2\left[\Phi(\vec{0}, t_L) - \frac{H_G^2(t_L)}{\dot{H}_G(t_L)}\left(\Phi(\vec{0}, t_L) + \frac{\Phi(\vec{0}, t_L)_{,t_L}}{H_G(t_L)}\right)\right]_{,r_G r_G} = \frac{5}{3}H_G^2 a_G^3 \delta_{l,0} . \tag{116}$$

Here we have inserted a subscript $_l$ to $\delta$ to make it more explicit that it represents a long wavelength fluctuation. The growth now depends on the effective curvature in two ways. First,

---

[12]We stress that we perform this approximation just in this appendix to make contact with former literature, but we do not do this same approximation in the main text, where the derivation is performed in full GR.

the growth in overdense regions is enhanced by a factor of $20\delta_l/21$. Furthermore, from (97) we obtain for the relation between local and global expansion $a_L(t_L) = (1 - \delta_l/3)a_G(t_L)$, and, at a given fixed proper time, thus we have to evaluate the local growth at an earlier (later) scale-factor for overdense (underdense) regions. This partially cancels the first dependence. Adding both contributions, the derivative of the growth rate with respect to the long wavelength density reads as

$$\left.\frac{\partial D}{\partial \delta_{l,0}}\right|_{a_G} = \left.\frac{\partial D}{\partial \Omega_K}\right|_{a_L} \left.\frac{\partial \Omega_K}{\partial \delta_{l,0}}\right|_{a_G} + \left.\frac{\partial D}{\partial a_L}\right|_{\Omega_K} \left.\frac{\partial a_L}{\partial \delta_{l,0}}\right|_{a_G} = \frac{20}{21}a_G^2 - \frac{1}{3}a_G^2 = \frac{13}{21}a_G^2 \ , \tag{117}$$

Thus, we finally have with $\delta_l(t) = a_G \delta_{l,0}$:

$$D(\delta_l \neq 0) = D_0 + \left.\frac{\partial D}{\partial \delta_{l,0}}\right|_0 \delta_{l,0} = a_G \left(1 + \frac{13}{21}\delta_l\right) \tag{118}$$

It turns out that the coupling strength of $13/21$ is a particlular property of the Einstein-de-Sitter Universe. In a more general $\Lambda$CDM Universe the coupling is less strong. Thus we will write the enhanced growth generally as $D = D_0(1 + \beta\delta_l)$ in the following. From the rescaling between the global and local Hubble rate we can derive the rescaling of the local mean density

$$\frac{H_L^2 + K/a^2}{H_G^2} = \frac{\bar{\rho}_L}{\bar{\rho}_G} \tag{119}$$

leading to

$$\bar{\rho}_L = \bar{\rho}_G(1 + \delta_l) \tag{120}$$

Since $\rho$ is a scalar, local and global density agree $\rho_G(x) = \rho_L(x)$ when evaluated at the same physical point. Given that in this approximation we are neglecting the difference between $t_L$ and $t_G$, we have

$$\delta_L(\vec{x}) = \frac{\rho(\vec{x})}{\bar{\rho}_L} - 1 \ , \quad \delta_G(\vec{x}) = \frac{\rho(\vec{x})}{\bar{\rho}_G} - 1 \quad \Rightarrow \quad \delta_G = (1 - \delta_l)(1 + \delta_L) - 1 \ . \tag{121}$$

Manipulating the last expression, we obtain

$$\delta_G(x) = \delta_{L,0}(1 + \beta\delta_l)(1 + \delta_l) + \delta_l = \delta_{L,0}[1 + (1 + \beta)\delta_l] + \delta_l = \delta_{L,0}\left(1 + \frac{34}{21}\delta_l\right) + \delta_l \ , \tag{122}$$

where in the last step we have assumed EDS Universe. Here, we accounted both for the excess growth in the local frame and for the rescaling of the local mean density, with respect to which the local overdensity is defined. The three point function between long and short modes thus reads

$$\langle \delta_{\mathrm{G,s}}(\vec{x})\delta_{\mathrm{G,s}}(\vec{x})\delta_l(\vec{x})\rangle = 2 \times \frac{34}{21}\sigma_s^2 P_l(k) \ . \tag{123}$$

## D.1 Correlators between Long and Short Modes

The coupling between long and short modes in the Newtonian regime can also be examined using perturbation theory (for a review see [39]). Standard perturbation theory solves the Newtonian fluid equations using a perturbative expansion in matter density and velocity divergence. In an Einstein de Sitter Universe, the second order contribution to the matter density field can be calculated as

$$\delta^{(2)}(\vec{k}) = \int \frac{d^3q}{(2\pi)^3} F_2(\vec{q}, \vec{k} - \vec{q}) \, \delta^{(1)}(\vec{q}) \, \delta^{(1)}(\vec{k} - \vec{q}) \tag{124}$$

where $\delta^{(1)}$ is the linearly evolved primordial density field and the second order mode coupling kernel is defined as

$$F_2(\vec{k}_1, \vec{k}_2) = \frac{5}{7} + \frac{1}{2} \frac{\vec{k}_1 \cdot \vec{k}_2}{k_1 k_2} \left( \frac{k_2}{k_1} + \frac{k_1}{k_2} \right) + \frac{2}{7} \frac{\left( \vec{k}_1 \cdot \vec{k}_2 \right)^2}{k_1^2 k_2^2} . \tag{125}$$

We can now apply (124) to the case where we have a Universe with short modes $\delta_s(\vec{k}_s)$ and a spherical symmetric monochromatic long mode $\delta_l(k_l)$. In this case we get for the matter field up to second order

$$\delta_s^{(2)}(\vec{k}_s) = \int \frac{d\Omega_l}{4\pi} F_2(\vec{k}_s, \vec{k}_l) \delta_s^{(1)}(\vec{k}_s) \delta_l^{(1)}(k_l) = \delta_s^{(1)}(\vec{k}_s) \left( 1 + \frac{34}{21} \delta_l^{(1)}(k_l) \right) , \tag{126}$$

where we neglected the coupling of the short and long modes with themselves.

The skewness of the density field at second order is

$$\langle \delta(\vec{x})^3 \rangle = 6 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} P(q) P(q') F_2(\vec{q}, \vec{q}') = 3 \times \frac{34}{21} \sigma^4 . \tag{127}$$

The prefactor 3 arises from the fact that all three density fields in $\langle \delta^3 \rangle$ can be expanded to second order. For the correlator between short and long modes we obtain

$$\langle \delta_s(\vec{x}) \delta_s(\vec{x}) \delta_l(\vec{x}) \rangle = 4 \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} P_s(q) P_l(q') F_2(\vec{q}, \vec{q}') = 2 \times \frac{34}{21} \sigma_s^2 P(k_l) , \tag{128}$$

where we assumed $P_l(\vec{q}) = (2\pi)^3 \delta^{(D)}(\vec{q} - \vec{k}) P(\vec{k})$, $k_l$ is the long wavelength and the prefactor 2 arises from the fact that now only two of the three fields can be expanded to second order. This is in perfect agreement with our result in (123).

# E   Spherical Collapse Dynamics

The collapse of a dark matter halo can be calculated considering a spherical overdensity within an otherwise homogeneous background Universe. In the standard calculation the background is assumed to be a flat matter-only Universe (aka Einstein-de-Sitter Universe). After reviewing the standard spherical collapse dynamics we extend the calculation to the case where the background Universe is curved. As we argued in the main text, this corresponds to the collapse in the presence of a long-wavelength mode. This procedure will offer us a way to match our General Relativistic definition of the bias with the standard Newtonian definition.

## E.1 Collapse in Flat FRW

According to Birkhoff's theorem, a spherically symmetric overdense region evolves as a closed FRW Universe, whose Friedmann equation reads as

$$H_C^2 = \left(\frac{\dot{a}_C}{a_C}\right)^2 = \frac{8\pi G \rho_C}{3 a_C^3} - \frac{K_C}{a_C^2} \ , \tag{129}$$

where the subscript $C$ is used to refer to the collapsing region. This collapsing region typically has the size of a dark matter halo and should not be mistaken for the local patch described in the main part of this paper, which can contain many of these collapsing regions. The time evolution of the scale factor of the closed patch can be parametrized by the cycloid solution

$$a_C = A_C \left(1 - \cos\theta\right) \ , \qquad t = B_C \left(\theta - \sin\theta\right) \ , \quad \text{with} \quad \theta \in [0, 2\pi] \ , \tag{130}$$

where we defined

$$A_C = \frac{4\pi G \rho_C}{3 K_C} \ , \quad \text{and} \quad B_C = \frac{4\pi G \rho_C}{3 K_C^{3/2}} \ . \tag{131}$$

The spherical overdense region described by this parametrization expands until $\theta = \pi$, then it turns around to collapse at $\theta = 2\pi$, corresponding to the collapse time $t_{coll} = 2\pi B_C$. Formally the expansion at the collapse time is zero, but physically one expects the region to form a virialized object at some time between turnaround and collapse. At early times $\theta \ll 1$ the parametric solution can be expanded as

$$a_C = A_C \frac{\theta^2}{2} \left(1 - \frac{\theta^2}{12} + \frac{\theta^4}{360} - \frac{\theta^6}{20160} + \dots\right) \ , \tag{132}$$

$$t = B_C \frac{\theta^3}{6} \left(1 - \frac{\theta^2}{20} + \frac{\theta^4}{840} - \frac{\theta^6}{60480} + \dots\right) \ . \tag{133}$$

Solving the above equations consistently up to order $\mathcal{O}(\theta^4)$ one obtains

$$a_C = A_C \frac{6^{2/3}}{2} \left(\frac{t}{B_C}\right)^{2/3} \left[1 - \frac{6^{2/3}}{20} \left(\frac{t}{B_C}\right)^{2/3}\right] \ . \tag{134}$$

The linear overdensity of the closed Universe collapsing at $t_{coll}$ is then given by the fractional deviation between the local and the background volume (described here by the respective expansion factors)

$$\delta(t; t_{coll}) = \frac{a_B^3}{a_C^3} - 1 = \frac{3}{5} \left(\frac{3\pi}{2}\right)^{2/3} \left(\frac{t}{t_{coll}}\right)^{2/3} = \frac{3}{5} \left(\frac{3\pi}{2}\right)^{2/3} \frac{1 + z_{coll}}{1 + z(t)} \ , \tag{135}$$

where we have used that the matter-only background Universe evolves according to $a_B \propto t^{2/3}$. Finally, one obtains the critical density for collapse at $z_{coll}$, linearly extrapolated to the present time $z(t_0) = 0$

$$\delta_c(z_{coll}) = \frac{3}{5} \left(\frac{3\pi}{2}\right)^{2/3} (1 + z_{coll}) \approx 1.686 \left(1 + z_{coll}\right) \ . \tag{136}$$

## E.2  Closed Background

We will now extend the above calculation to the case, where the collapsing region resides in a curved background Universe following [44] and [45]. We will consider the case of an overdense, closed background Universe and note that the open background can be treated analogously. Furthermore, we will restrict ourselves to a background Universe without a dark energy component, such that only matter and curvature contribute to the energy budget. This closed background Universe is parametrized as

$$a_B = A_B \left(1 - \cos \eta \right) \qquad\qquad t = B_B \left(\eta - \sin \eta \right) \tag{137}$$

with $\eta \in [0, 2\pi]$ and the parameters

$$A_B = \frac{4\pi G \rho_B}{3 K_B}, \quad \text{and} \quad B_B = \frac{4\pi G \rho_B}{3 K_B^{3/2}}. \tag{138}$$

This curved background can now be identified with the effective curved patch describing a long wavelength fluctuation. We study the evolution of a collapsing spherical overdensity, expanding the parametric solutions for both the background (137) and the collapsing region (130) at early times. This means that we restrict ourselves to treat the curvature of the background at linear order. The linear density contrast then scales as

$$\delta = \frac{a_C^3}{a_B^3} - 1 = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3}\left[\left(\frac{1}{t_{\rm coll}}\right)^{2/3} - \left(\frac{1}{t_\Omega}\right)^{2/3}\right] t^{2/3}, \tag{139}$$

where $t_\Omega = 2\pi B$. For $K \to 0$ we have $t_\Omega \to \infty$ and thus we recover the EDS result shown above. The overdensity for an object that collapses at $t_{\rm coll}$, linearly extrapolated to present time thus reads as

$$\frac{\delta_{\rm c}(z_{coll})}{1 + z_{\rm coll}} = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3}\left[1 - \left(\frac{t_0}{t_\Omega}\right)^{2/3}\left(\frac{t_{coll}}{t_0}\right)^{2/3}\right]. \tag{140}$$

We can now write down the collapse time for the background Universe.

$$t_\Omega = 2\pi B_B = \frac{\pi \Omega_{m,B}}{H_0(\Omega_{m,B} - 1)^{3/2}} = \frac{\pi (1 - \Omega_{K,B})}{H_0(-\Omega_{K,B})^{3/2}} \tag{141}$$

Using $(t_0/t_{\rm coll})^{2/3} = 1 + z_{coll}$ and $t_0 \approx 2/(3H_0)$ the overdensity of the collapsing region can be rewritten as

$$\frac{\delta_{\rm c}(z_{coll})}{1 + z_{\rm coll}} = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3}\left[1 + \left(\frac{2}{3\pi}\right)^{2/3}\frac{\Omega_{K,B}}{(1 - \Omega_{K,B})^{2/3}}\frac{1}{1 + z_{coll}}\right]. \tag{142}$$

We are now going to consider the case where the curvature of the background Universe can be described by a long wavelength fluctuation with present day amplitude $\delta_{l,B}$. In this case we obtain for the density parameters of the background Universe

$$\Omega_{m,B} = \frac{1 + \delta_{l,B}}{(1 - \delta_{l,B}/3)^2} = 1 + \frac{5}{3}\delta_{l,B} \quad \Rightarrow \quad \Omega_{K,B} = -\frac{5}{3}\delta_{l,B}. \tag{143}$$

For the overdensity of a perturbation that collapses at $z_{\text{coll}}$ linearly extrapolated to the present day we obtain

$$\delta_c(z_{coll}) \approx 1.686\,(1 + z_{coll}) - \delta_{l,0} = \delta_c(\Omega_{K,B} = 0) - \delta_{l,B} \ . \tag{144}$$

This result is of course very intuitive and it allows us to explicitly verify that our General Relativistic definition of the bias agrees with the standard Newtonian one.

# F    Perturbed Geodesic Parameters

In this appendix we will provide the essence of cosmological perturbation theory and expalain the gauge choices used in this paper. Then we will quickly review the most important formulae required for the mapping to observables before we conclude by specialising our result for the observed overdensity to the case where the matter distribution itself is the tracer.

## F.1    Gauge Transformations

The most general perturbed metric for a flat Universe reads as [47]

$$ds^2 = -(1 + 2A)dt^2 - 2aB_i dx^i dt + a^2 \left[(1 + 2D)\delta_{ij} + E_{ij}\right] dx^i dx^j \ . \tag{145}$$

We will restrict ourselves to scalar modes $B_i = BQ_i^{(0)}$ and $E_{ij} = EQ_{ij}^{(0)}$, where $Q^{(0)}$ is the scalar eigenmode of the Laplacian. Here we perform the scalar-vector-tensor decomposition in $k$-space, where $Q^{(0)} = \exp\left[i\vec{k} \cdot \vec{x}\right]$. We consider a Universe filled with dark matter plus dark energy and neglect anisotropic stress and pressure perturbations.

A gauge transformation corresponds to a change in spatial position $x^i$ and comoving time $ad\tau = dt$

$$\tilde{x}^i = x^i + L^i \qquad\qquad\qquad \tilde{\tau} = \tau + T \tag{146}$$

under such a transformation the metric perturbations transform as

$$\tilde{A} = A - a\dot{T} - aHT \qquad\qquad \tilde{D} = D - \frac{k}{3}L - aHT \tag{147}$$
$$\tilde{B} = B + a\dot{L} + kT \qquad\qquad \tilde{E}_{ij} = E_{ij} + kT$$

and the components of the energy momentum tensor transform as

$$\tilde{\delta} = \delta + 3aHT \qquad\qquad\qquad \tilde{v} = v + a\dot{L} \ , \tag{148}$$

where $v_i = vQ_i^{(0)}$. Gauge invariance refers to the fact that certain combinations of metric and energy momentum perturbations are invariant under a change of coordinates (146), i.e., the numerical value of a quantity does not change. Gauge invariance is necessary, but not sufficient, for observability.

We will consider two gauges

1. Newtonian Gauge

   Newtonian gauge is defined by $B = E = 0$ setting $A = \Phi$ and $D = -\Psi$.

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(1 - 2\Psi)d\vec{x}^2 \tag{149}$$

   Neglecting anisotropic stress we have $\Phi = \Psi$. For the Einstein equations we have

$$-k^2\Phi - 3a^2H^2\left(\Phi + \frac{\dot{\Phi}}{H}\right) = 4\pi Ga^2\bar{\rho}\delta^{(N)} \tag{150}$$

$$aH\left(\Phi + \frac{\dot{\Phi}}{H}\right) = 4\pi Ga^2\left(\bar{\rho} + \bar{p}\right)\frac{v^{(N)}}{k} = -a^2\dot{H}\frac{v^{(N)}}{k} \tag{151}$$

2. Comoving Gauge

   In comoving gauge we set $E = 0$ and $\delta T_i^0 = 0$ corresponding to $v_i = B_i$. Setting $A = \xi$ and $D = \zeta$ leads to the metric

$$ds^2 = -(1 + 2\xi)dt^2 + av_i dx^i dt + a^2(t)(1 + 2\zeta)d\vec{x}^2 \tag{152}$$

   then the Einstein equations read as

$$k^2\left(\zeta + aH\frac{v^{(com)}}{k}\right) = 4\pi Ga^2\bar{\rho}_m\delta^{(com)} \tag{153}$$

$$H\xi - \dot{\zeta} = 0 , \tag{154}$$

   where $v^{(com)} = v^{(N)}$. One can show that on scales larger than the sound horizon $\zeta$ is constant. Using $\dot{\zeta} = 0$ we see that the lapse function $\xi$ vanishes in pressureless media and thus the comoving gauge is also synchronous, i.e., proper time agrees with the coordinate time. Using

$$T_{N\to com} = \frac{v^{(N)} - B^{(N)}}{k} = \frac{v^{(N)}}{k} \tag{155}$$

   the overdensities in the Newtonian and comoving gauge are related by

$$\delta^{(com)} = \delta^{(N)} - a\dot{\bar{\rho}}T_{N\to com} = \delta^{(N)} + 3aH\frac{v^{(N)}}{k} \tag{156}$$

   For the spatial metric perturbations we have

$$\zeta = -\Phi - aHT_{N\to com} = -\Phi - aH\frac{v^{(N)}}{k} = -\Phi + \frac{H^2}{\dot{H}}\left(\Phi + \frac{\dot{\Phi}}{H}\right) \tag{157}$$

   In synchronous gauge $A = B = 0$ there are no sources in the equation of motion for the velocity of stress free matter, i.e., if it was at rest initially it will remain so for all times [48]. In this case the density perturbation in comoving and synchronous gauge agree $\delta^{(com)} = \delta^{(syn)}$. The CMBFAST Boltzmann code [23] is providing the synchronous gauge transfer function and can thus be used to infer the transfer function for the comoving gauge density perturbation.

Combining the Einstein equations in comoving and Newtonian gauge we have

$$-k^2\Phi = 4\pi Ga^2\delta^{(com)} , \tag{158}$$

which is valid on *all* scales.

## F.2 Volume Distortion & Observed Redshifts

Here we explain the symbols used in eq. (50) for the reader's convenience (see [1] for a detailed explanation and derivation):

$$\mathcal{J} = \delta V/V = - \Phi + v^i e_i - (1+z)\frac{d}{dz}\delta z_{G\to z} - 2\frac{1+z}{Hr}\delta z_{G\to z} - \delta z_{G\to z}$$

$$-2\kappa + \frac{1+z}{H}\frac{dH}{dz}\delta z_{G\to z} + 2\frac{\delta r}{r} \tag{159}$$

$$= - \Phi + \left[\frac{d\ln H}{d\ln(1+z)} - 1 - 2\frac{1+z}{Hr_s}\right]\left[v^i e_i - \Phi - 2\int_0^{r_s} dr\, a\dot\Phi\right]$$

$$+\frac{4}{r_s}\int_0^{r_s} dr\,\Phi - 2\int_0^{r_s} dr\,\frac{r_s-r}{r r_s}\hat\nabla\Phi + \frac{1}{H}\left[\dot\Phi - \frac{1}{a}\frac{\partial v_i e^i}{\partial r}\right].$$

In the evaluation of the above expression we used that the total derivative is given by $d/dz = H^{-1}d/dr = -(\partial_o - e^i\partial_i) = -(\partial_0 - \partial_r)$ and that the velocity follows the evolution equation

$$\dot v_i + Hv_i = -\frac{\Phi_{,i}}{a}. \tag{160}$$

The perturbation to the redshift of the source $\delta z_{G\to z}$, is given by the relationship

$$\delta z_{G\to z} = aH\delta\tau_o + \left[v_i e^i - \Phi\right]_0^s - 2\int_0^{r_s} a\dot\Phi\, dr. \tag{161}$$

Here the four velocity of the source is given by $u^\alpha = a^{-1}\big((1+\Phi), v^i\big)$ and $e^i$ is the photon propagation direction as seen from the observer, $r$ is the comoving line-of-sight distance, $r_s$ is the comoving line-of-sight distance of the source. $\delta\tau_o$ is the perturbation to the conformal time at the time of observation, and it is just a monopole term that cancels when measuring fluctuations. $\delta r$ is the radial displacement, given by

$$\delta r = \delta\tau_0 + 2\int_0^{r_s} dr\,\Phi. \tag{162}$$

The deflection of the photons on their way form the source galaxy to the observer can be quantified as

$$\delta\theta = 2\int_0^{r_s} dr\left(\frac{r_s-r}{r\,r_s}\right)\Phi_{,\theta}, \tag{163}$$

$$\delta\phi = 2\int_0^{r_s} dr\left(\frac{r_s-r}{r\,r_s\sin\theta}\right)\Phi_{,\phi}.$$

The latter can be combined to calculate the distortion of the solid angle as quantified by the convergence $\kappa$

$$\kappa = 2\int_0^{r_s} dr\left(\frac{r_s-r}{r\,r_s}\right)\hat\nabla^2\Phi, \tag{164}$$

where $\hat\nabla$ is the differential operator on the two dimensional unit sphere. In case the survey is not volume limited but rather flux limited, we need to replace $\mathcal{J}$ with

$$\mathcal{J} \rightarrow \mathcal{J} - 5p\,\delta\mathcal{D}_L, \tag{165}$$

42

where $\delta \mathcal{D}_L$ is the perturbation to the luminosity distance, which is given by

$$
\begin{aligned}
\frac{\delta \mathcal{D}_L(z)}{D_L(z)} = {}& 1 + v_i e^i - \Phi_s - \frac{1+z_s}{H_s r_s} \delta z_{G \to z} + \left( \mathcal{H}_o + \frac{1}{r_s} \right) \delta \tau_o \\
& + 2 \int_0^{r_s} dr \left[ \frac{\Phi}{r_s} - \frac{r}{r_s} a \dot{\Phi} + \frac{(r_s - r)r}{2r_s} \left( \nabla^2 \Phi - a \frac{d(a\dot{\Phi})}{dt} + 2a\dot{\Phi}_{,i} e^i \right) \right] , \quad (166)
\end{aligned}
$$

where $D_L(z) = (1+z)r(z)$ is the unperturbed luminosity distance and $p$ is the slope of the luminosity function.

Using $dz = -H(1+z)dt$ we can write in a general and in Newtonian gauge

$$
\begin{aligned}
\delta z_{G \to L} = {}& \frac{z_G(t_L) - z_G(t_G)}{1 + z_G(t_G)} = -H(t_G)(t_L - t_G) = -H \int_0^{t_G} dt A \\
= {}& -H \int_0^{t_G} dt \; \Phi = -aH \frac{v^{(N)}}{k} , \quad (167)
\end{aligned}
$$

$$
\begin{aligned}
\delta z_{G \to z} = {}& \frac{z - z_G(t_G)}{1 + z_G} = \left[ (v_i - B_i) e^i - A \right]_o^s - \int_0^{r_s} dr \left[ a \left( \dot{A} - \dot{D} \right) - \left( B_{i,j} + a\dot{E}_{ij} \right) e^i e^j \right] \\
= {}& \left[ v_i e^i - \Phi \right]_o^s - 2 \int_0^{r_s} dr \; a\dot{\Phi} . \quad (168)
\end{aligned}
$$

Under a gauge transformation (146) we have

$$
\widetilde{\delta z}_{G \to L} = \delta z_{G \to L} + aHT , \qquad\qquad \widetilde{\delta z}_{G \to z} = \delta z_{G \to z} + aHT . \quad (169)
$$

Thus $\delta z_{G \to L} - \delta z_{G \to z}$ does not change under the gauge transformation and is gauge invariant. For the evaluation of the full expression in eq. (61) we first transform all the quantities to $k$-space. First, we consider the line of sight projection of the velocity

$$
v_i e^i = \int \frac{d^3 k}{(2\pi)^3} \left( -i\mu v^{(N)}(\vec{k}) \right) Q^{(0)}(\vec{k}) , \quad (170)
$$

where $\mu = \vec{x} \cdot \vec{k}/(xk)$ is the cosine between the $k$-mode and the line of sight. For the redshift space distortion term we have then

$$
\partial_r v_i e^i = n^j \partial_j e^i v_i = \int \frac{d^3 k}{(2\pi)^3} \left( \mu^2 v^{(N)}(\vec{k}) \right) Q^{(0)}(\vec{k}) . \quad (171)
$$

Thus the volume distortion term reads as

$$
\mathcal{J}(\vec{k}) = -\Phi(\vec{k}) + \mathcal{A}(z) \left( -i\mu v^{(N)}(\vec{k}) - \Phi(\vec{k}) \right) + \left( f(z) - 1 \right) \Phi(\vec{k}) - \mu^2 \frac{k}{aH} v^{(N)}(\vec{k}) . \quad (172)
$$

The full observed density perturbation is the sum of the latter and the perturbation in the proper number density of tracers

$$
\delta_p(\vec{k}) = b\, \delta^{(com)}(\vec{k}) + b_\zeta \zeta(\vec{k}) + \mathcal{B}(z) \left( -\frac{aH}{k} v^{(N)}(\vec{k}) - i\mu v^{(N)}(\vec{k}) - \Phi(\vec{k}) \right) , \quad (173)
$$

43

$$\delta_{obs}(\vec{k}) = \left\{ \left[ -2 - \mathcal{A}(z) + f(z) + f(z)\frac{\mathcal{B}(z)}{\beta(z)} + B(z) - b_\zeta \left( 1 - \frac{f(z)}{\beta(z)} \right) \right. \right.$$
$$\left. \left. + \left( \mu^2 \frac{f(z)}{\beta(z)} - \frac{b}{\alpha(z)} \right) \left( \frac{k}{aH} \right)^2 \right] + i \left[ (\mathcal{A}(z) - \mathcal{B}(z))\mu \frac{f(z)}{\beta(z)} \frac{k}{aH} \right] \right\} \Phi(\vec{k}) , \qquad (174)$$

where we related the density and velocity perturbations to the Newtonian gauge metric perturbation

$$\delta^{(com)}(\vec{k}) = -\left( \frac{k}{aH} \right)^2 \frac{H^2}{4\pi G\bar{\rho}} \Phi(\vec{k}) , \qquad\qquad v(\vec{k}) = f\frac{k}{aH}\frac{H^2}{\dot{H}} \Phi(\vec{k}) , \qquad (175)$$

and introduced the auxiliary functions

$$\mathcal{A}(z) = \frac{d\log H}{d\log(1+z)} - 1 - 2\frac{1+z}{Hr_s} = \frac{3}{2}\frac{(1+z)^2}{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}} - 1 - 2\frac{(1+z)c}{r_s H} , \qquad (176)$$

$$\beta = \frac{\dot{H}}{H^2} , \qquad\qquad \alpha = \frac{4\pi G\bar{\rho}}{H^2} , \qquad\qquad \mathcal{B}(z) = \frac{\partial\log\bar{n}_p}{\partial\log(1+z)} .$$

The power spectrum is then given by $(2\pi)^3\delta^{(D)}(\vec{k} + \vec{k}')P_{obs}(k) = \left\langle \delta_{obs}(\vec{k})\delta^*_{obs}(\vec{k}') \right\rangle.$

## F.3 Matter as Tracer

For matter we have $n_p(t_L; \Omega_K) = \bar{\rho}(t_L)\left(1 + D(t_L)\delta_0^{(com)}\right)$. This can be seen in two ways: firstly, in synchronous slicings the proper time and the coordinate time agree, thus the matter overdensity in the local frame must agree with the matter overdensity in comoving gauge. Also, we saw above in eq. (102), that the local matter density is related to the global one by $\bar{\rho}_L = \bar{\rho}_G(1 + D(t_L)\delta_{com,0})$. Using that $\Omega_{K,0} \propto \delta_{com,0}$ and that $\partial\log\bar{\rho}/\partial\log(1+z) = 3$ we have

$$\delta_{obs}(z, \theta, \phi) = \delta^{(com)}(z_G) + 3\delta z_{G\to L} - 3\delta z_{G\to z} + \mathcal{J} , \qquad (177)$$

where the $\delta z$'s are in a general, yet unspecified gauge.
For the transformation from a comoving to general gauge we have for the densities

$$\delta^{(gen)} = \delta^{(com)} + 3aHT_{com\to gen} . \qquad (178)$$

The integral entering into $\delta z_{G\to L}$ transforms as

$$3\delta z_{G\to L}^{(gen)} = -3H \int dt A^{(gen)} = -3H \int dt A^{(com)} + 3aHT_{com\to gen} = 3aHT_{com\to gen} = \delta^{(gen)} - \delta^{(com)} , \qquad (179)$$

where we used that $A^{(com)} = 0$ and solved eq. (178) for $T_{com\to gen}$. Evaluating $\delta_{obs}$ in the general gauge we obtain

$$\delta_{obs}(z, \theta, \phi) = \delta^{(com)} - 3H \int dt A^{(gen)} - 3\delta z_{G\to z}^{(gen)} + \mathcal{J} = \delta^{(gen)} - 3\delta z_{G\to z}^{(gen)} + \mathcal{J}, \qquad (180)$$

44

where $\delta^{(gen)}$ and $\delta z_{G\to L}^{(gen)}$ are the matter overdensity and the redshift lapse in a general gauge. This expression agrees with $\delta_{n_p} = \delta^{(gen)} - 3\delta z_{G\to z}^{(gen)}$ in [1]. In the follow up paper, [13] used a local bias in the matter density at the observed redshift $\delta^{(gen)} - 3\delta z_{G\to z}^{(gen)}$. This means that in contrast to our approach the bias factor $b$ is also multplying the redshift lapse terms between global, local and observed redshift, i.e. the evolution of the sample is fixed to be $\partial \log \bar{n}_p / \partial \log(1+z) = 3b$, while for a typical quasar sample this number can vary in a much wider range depending on the redshift distribution of the sample.

# References

[1] J. Yoo, A. L. Fitzpatrick and M. Zaldarriaga, "A New Perspective on Galaxy Clustering as a Cosmological Probe: General Relativistic Effects," Phys. Rev. D **80** (2009) 083514 [arXiv:0907.0707 [astro-ph.CO]].

[2] N. Dalal, O. Dore, D. Huterer and A. Shirokov, "The imprints of primordial non-gaussianities on large-scale structure: scale dependent bias and abundance of virialized objects," Phys. Rev. D **77** (2008) 123514 [arXiv:0710.4560 [astro-ph]].

[3] A. Slosar, C. Hirata, U. Seljak, S. Ho and N. Padmanabhan, "Constraints on local primordial non-Gaussianity from large scale structure," JCAP **0808** (2008) 031 [arXiv:0805.3580 [astro-ph]].

[4] L. Senatore and M. Zaldarriaga, "The Effective Field Theory of Multifield Inflation," arXiv:1009.2093 [hep-th].

[5] T. Baldauf, U. Seljak and L. Senatore, "Primordial non-Gaussianity in the Bispectrum of the Halo Density Field," JCAP **1104** (2011) 006 [arXiv:1011.1513 [astro-ph.CO]].

[6] E. Fermi, "Sopra i fenomeni che avvengono in vicinanza di una linea oraria," Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. 31,184-187, 306-309, 1922. F. K. Manasse and C. W. Misner, "Fermi Normal Coordinates and Some Basic Concepts in Differential Geometry," J. Math. Phys. 4, 735 (1963).

[7] G. Tormen and E. Bertschinger, "Adding Long Wavelength Modes to an $N$-Body Simulation," arXiv:astro-ph/9512131.

[8] S. Cole, "Adding Long-Wavelength Power to N-body Simulations," arXiv:astro-ph/9604046.

[9] M. D. Schneider, S. Cole, C. S. Frenk and I. Szapudi, "Fast generation of ensembles of cosmological N-body simulations via mode-resampling," arXiv:1103.2767 [astro-ph.CO].

[10] M. Sasaki, "The Magnitude - Redshift relation in a perturbed Friedmann Universe," Mon. Not. Roy. Astron. Soc. **228**, 653-669 (1987).

[11] S. Dodelson, F. Schmidt and A. Vallinotto, "Universal Weak Lensing Distortion of Cosmological Correlation Functions," Phys. Rev. D **78**, 043508 (2008) [arXiv:0806.0331 [astro-ph]].

[12] D. Wands and A. Slosar, "Scale-dependent bias from primordial non-Gaussianity in general relativity," Phys. Rev. D **79** (2009) 123507 [arXiv:0902.1084 [astro-ph.CO]].

[13] J. Yoo, "General Relativistic Description of the Observed Galaxy Power Spectrum: Do We Understand What We Measure?," Phys. Rev. **D82**, 083508 (2010). [arXiv:1009.3021 [astro-ph.CO]].

[14] N. Bartolo, S. Matarrese, A. Riotto, "Relativistic effects and primordial non-Gaussianity in the galaxy bias," JCAP **1104**, 011 (2011). [arXiv:1011.4374 [astro-ph.CO]].

[15] E. Komatsu *et al.*, "Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation," arXiv:1001.4538 [astro-ph.CO].

[16] L. Senatore, K. M. Smith and M. Zaldarriaga, "Non-Gaussianities in Single Field Inflation and their Optimal Limits from the WMAP 5-year Data," JCAP **1001** (2010) 028 [arXiv:0905.3746 [astro-ph.CO]].

[17] P. Creminelli, M. A. Luty, A. Nicolis and L. Senatore, "Starting the Universe: Stable violation of the null energy condition and non-standard cosmologies," JHEP **0612** (2006) 080 [arXiv:hep-th/0606090]. P. Creminelli, G. D'Amico, J. Norena and F. Vernizzi, "The Effective Theory of Quintessence: the $w < -1$ Side Unveiled," JCAP **0902** (2009) 018 [arXiv:0811.0827 [astro-ph]]. P. Creminelli, G. D'Amico, J. Norena, L. Senatore and F. Vernizzi, "Spherical collapse in quintessence models with zero speed of sound," JCAP **1003** (2010) 027 [arXiv:0911.2701 [astro-ph.CO]].

[18] N. E. Chisari and M. Zaldarriaga, "On the connection between Newtonian simulations and General Relativity," arXiv:1101.3555 [astro-ph.CO].

[19] A. L. Fitzpatrick, L. Senatore and M. Zaldarriaga, "Contributions to the Dark Matter 3-Pt Function from the Radiation Era," JCAP **1005** (2010) 004 [arXiv:0902.2814 [astro-ph.CO]].

[20] C. Cheung, P. Creminelli, A. L. Fitzpatrick, J. Kaplan and L. Senatore, "The Effective Field Theory of Inflation," JHEP **0803** (2008) 014 [arXiv:0709.0293 [hep-th]].

[21] G. Tormen and E. Bertschinger, "Adding Long Wavelength Modes to an $N$-Body Simulation," arXiv:astro-ph/9512131.

[22] S. Cole, "Adding Long-Wavelength Power to N-body Simulations," arXiv:astro-ph/9604046.

[23] U. Seljak and M. Zaldarriaga, "A Line of Sight Approach to Cosmic Microwave Background Anisotropies," Astrophys. J. **469** (1996) 437 [arXiv:astro-ph/9603033].

[24] A. Lewis, A. Challinor and A. Lasenby, "Efficient Computation of CMB anisotropies in closed FRW models," Astrophys. J. **538** (2000) 473 [arXiv:astro-ph/9911177].

[25] A. Gangui, F. Lucchin, S. Matarrese and S. Mollerach, "The Three Point Correlation Function Of The Cosmic Microwave Background In Inflationary Models," Astrophys. J. **430** (1994) 447 [arXiv:astro-ph/9312033].

[26] D. H. Lyth, C. Ungarelli and D. Wands, "The primordial density perturbation in the curvaton scenario," Phys. Rev. D **67**, 023503 (2003) [arXiv:astro-ph/0208055].

[27] M. Zaldarriaga, "Non-Gaussianities in models with a varying inflaton decay rate," Phys. Rev. D **69**, 043508 (2004) [arXiv:astro-ph/0306006].

[28] P. Creminelli and L. Senatore, "A smooth bouncing cosmology with scale invariant spectrum," JCAP **0711** (2007) 010 [arXiv:hep-th/0702165].

[29] X. Chen and Y. Wang, "Quasi-Single Field Inflation and Non-Gaussianities," JCAP **1004** (2010) 027 [arXiv:0911.3380 [hep-th]].

[30] M. Bruni, R. Crittenden, K. Koyama, R. Maartens, C. Pitrou and D. Wands, "Disentangling non-Gaussianity, bias and GR effects in the galaxy distribution," arXiv:1106.3999 [astro-ph.CO].

[31] E. Silverstein and A. Westphal, "Monodromy in the CMB: Gravity Waves and String Inflation," Phys. Rev. D **78** (2008) 106003 [arXiv:0803.3085 [hep-th]].

[32] L. McAllister, E. Silverstein and A. Westphal, "Gravity Waves and Linear Inflation from Axion Monodromy," arXiv:0808.0706 [hep-th].

[33] D. Green, B. Horn, L. Senatore and E. Silverstein, "Trapped Inflation," Phys. Rev. D **80** (2009) 063533 [arXiv:0902.1006 [hep-th]].

[34] N. Barnaby, Z. Huang, L. Kofman and D. Pogosyan, "Cosmological Fluctuations from Infra-Red Cascading During Inflation," Phys. Rev. D **80** (2009) 043501 [arXiv:0902.0615 [hep-th]].

[35] R. Flauger, L. McAllister, E. Pajer, A. Westphal and G. Xu, "Oscillations in the CMB from Axion Monodromy Inflation," arXiv:0907.2916 [hep-th].R. Flauger and E. Pajer, "Resonant Non-Gaussianity," arXiv:1002.0833 [hep-th].

[36] X. Chen, R. Easther and E. A. Lim, "Large Non-Gaussianities in Single Field Inflation," JCAP **0706** (2007) 023 [arXiv:astro-ph/0611645].

[37] C. M. Hirata and U. Seljak, "Analyzing weak lensing of the cosmic microwave background using the likelihood function," Phys. Rev. D **67** (2003) 043001 [arXiv:astro-ph/0209489].

[38] C. M. Hirata and U. Seljak, "Reconstruction of lensing from the cosmic microwave background polarization," Phys. Rev. D **68** (2003) 083002 [arXiv:astro-ph/0306354].

[39] F. Bernardeau, S. Colombi, E. Gaztanaga and R. Scoccimarro, "Large-scale structure of the Universe and cosmological perturbation theory," Phys. Rept. **367** (2002) 1 [arXiv:astro-ph/0112551].

[40] A. Challinor and A. Lewis, "The linear power spectrum of observed source number counts," arXiv:1105.5292 [astro-ph.CO].

[41] C. Bonvin and R. Durrer, "What galaxy surveys really measure," arXiv:1105.5280 [astro-ph.CO].

[42] D. Jeong, F. Schmidt, C. M. Hirata, "Large-scale clustering of galaxies in general relativity," [arXiv:1107.5427 [astro-ph.CO]].

[43] D. Seery and J. E. Lidsey, "Primordial non-Gaussianities in single field inflation," JCAP **0506** (2005) 003 [arXiv:astro-ph/0503692]. X. Chen, M. x. Huang, S. Kachru and G. Shiu, "Observational signatures and non-Gaussianities of general single field inflation," JCAP **0701** (2007) 002 [arXiv:hep-th/0605045]. L. Boubekeur, P. Creminelli, J. Norena and F. Vernizzi, "Action approach to cosmological perturbations: the 2nd order metric in matter dominance," JCAP **0808** (2008) 028 [arXiv:0806.1016 [astro-ph]].

[44] Peebles, P. J. E. 1980. *The large-scale structure of the Universe.* Princeton University Press, 1980. 435 p. .

[45] C. G. Lacey and S. Cole, "Merger rates in hierarchical models of galaxy formation," Mon. Not. Roy. Astron. Soc. **262**, 627 (1993).

[46] F. I. Cooperstock, V. Faraoni and D. N. Vollick, "The influence of the cosmological expansion on local systems," Astrophys. J. **503**, 61 (1998) [arXiv:astro-ph/9803097].

[47] W. Hu, "Lecture Notes on CMB Theory: From Nucleosynthesis to Recombination," arXiv:0802.3688 [astro-ph].

[48] C. P. Ma and E. Bertschinger, "Cosmological perturbation theory in the synchronous and conformal Newtonian gauges," Astrophys. J. **455**, 7 (1995) [arXiv:astro-ph/9506072].